Constrained Quantization on Symplectic Manifolds and Quantum Distribution Functions

G. Jorjadze

Dept. of Theoretical Physics
Tbilisi Mathematical Institute
M.Aleksidze 1, 380093, Tbilisi
GEORGIA
e-mail: jorj@ictp.trieste.it
and jorj@imath.acnet.ge

Abstract

A quantization scheme based on the extension of phase space with application of constrained quantization technic is considered. The obtained method is similar to the geometric quantization. For constrained systems the problem of scalar product on the reduced Hilbert space is investigated and possible solution of this problem is done. Generalization of the Gupta-Bleuler like conditions is done by the minimization of quadratic fluctuations of quantum constraints. The scheme for the construction of generalized coherent states is considered and relation with Berezin quantization is found. The quantum distribution functions are introduced and their physical interpretation is discussed.

Introduction

It is well known that the standard canonical quantization is not the universal method for the quantization of Hamiltonian systems. Actually this method is applicable only for the systems with a phase space having the cotangent bundle structure. For the generalization of canonical quantization different methods were developed and the geometric quantization [1] is accepted as the most general one.

In [2] it was proposed a quantization scheme based on the extension of phase space with further application of constrained quantization method [3]. The obtained quantization turned out very similar to the geometric one. The present work is the continuation of the activity started in [2].

The similar method with extension of phase space was introduced in [4a], where for the quantization of constrained extended system the BFV (BRST) quantization was used. In [4a] one can find also a wide variety of references to different quantization methods and their short analyses. Among other recent papers, which also use some extension procedure, it should be noted [4b] and [4c].

The present paper is organized as follows:

In Section 1 the extended system is introduced. The phase space of the extended system is a cotangent bundle over the initial symplectic manifold $\mathcal{M}$. For the quantization of the extended system the canonical method is used and the pre-quantization operators arise as the result of some natural operator ordering.

In Section 2 for the extended system the certain constrained surface $\Phi_{f_k} = 0$ is introduced. The constraint functions Φ_{f_k} characterized by some complete set of observables f_k $(k = 1, ...2N)$ of the initial system and they form the set of the second class constraints. Further, the constraint operators $\hat{\Phi}_f$ are introduced and for the restriction of the extended quantum system the Dirac's $(\hat{\Phi}_f|\Psi\rangle = 0)$ and the Gupta-Bleuler like $((\hat{\Phi}_f + i\epsilon\hat{\Phi}_g)|\Psi\rangle = 0)$ conditions are used. Certainly, the Dirac's condition are used only for the half of commuting to each other constraints and the same number of complex conditions is used in Gupta-Bleuler case too. Here, the standard problems of constraint quantization arise and in Appendixes A and B the possible solution of these problems is considered. In particular, in Appendix A the scalar product problem of physical states is investigated. For the solution of this problem the limiting procedure $(\epsilon \to 0)$ with normalized physical states is used.

In Section 3, illustrating the quantization scheme described above, we consider two examples. The first one is a quantization on the plane and the second one on the cylinder.

In Section 4 we generalize the Gupta-Bleuler like conditions. For this we use the minimization of quadratic fluctuations of quantum constraints. Technical part of this method is described in Appendix C. The obtained condition contains the constraint operators in second order, and for the physical wave functions they are elliptic type equation on the phase space.

In Section 5 we introduce the general coherent states, which are related with some complete set of observables. The coherent states are constructed as the functions on the phase space and, at the same time, they are parameterized by the points of the

phase space. The coherent states form the over complete set of states and have some interesting properties. In particular, they minimize uncertainties of observables just they are related to. At the end of the section we construct special coherent states on the cylinder and study their behavior in the limit when the squeezing parameter tends to zero. In this limit we get the eigenstates of the angular-momentum.

In Section 6 we introduce the quantum distribution functions as the square of the modulus of physical wave functions. We get some smooth distributions on the phase space and these functions satisfy some elliptic type equation. This equation specifies the distribution functions for the pure states. The generalization for mixed states is done as the convex combination of pure ones. There are different classes of quantum distributions functions and each class is related to a certain complete set of observables of the system in consideration. We discuss the physical interpretation of the introduced distribution functions. Namely, we interpret them as the distributions obtained in the experiment with simultaneous measurement of observables which define the given class. At the end of the paper we discuss the possibility for the formulation of quantum mechanics in terms of quantum distribution functions without referring to the Hilbert space and the operator formalism.

1 Quantization on a Cotangent Bundle

We start with an introduction of some standard notations of the Hamiltonian dynamics (see for example [1]).

The phase space of a classical system is a symplectic manifold $\mathcal{M}$ and ξ^k, ($k = 1, ..., 2N$) are some local coordinates on $\mathcal{M}$. For simplicity, the symplectic form $\omega = 1/2\ \omega_{kl}(\xi)d\xi^k \wedge d\xi^l$ is assumed to be exact: $\omega = d\theta$. Thus, $\omega_{kl} = \partial_k\theta_l - \partial_l\theta_k$, where $\theta_k(\xi)$ are components of a 1-form $\theta = \theta_k(\xi)d\xi^k$.

Observables are smooth real functions on $\mathcal{M}$, and the set of all observables $\mathcal{O}(\mathcal{M})$ has the natural Poisson-Lie structure.

The Hamiltonian vector field constructed for an observable $f(\xi)$ is given by

$$V_f = \mathcal{V}_f^k\partial_k, \quad \text{with} \quad \mathcal{V}_f^k = \omega^{kl}\partial_l f \tag{1.1}$$

where ω^{kl} is the inverse (to ω_{kl}) matrix: $\omega^{ij}\omega_{jk} = \delta_k^i$. This field generates one-parameter family of local canonical transformations.

The Poisson bracket of two observables f and g is defined by

$$\{f, g\} \equiv 2\omega(V_f, V_g) = -\partial_k f \omega^{kl}\partial_l g \tag{1.2}$$

and for global coordinates we have

$$\{\xi^k, \xi^l\} = -\omega^{kl}(\xi) \tag{1.2'}$$

The Hamilton function $H = H(\xi)$ generates the dynamics of a system through the Hamilton's equations

$$\dot\xi^i = \mathcal{V}_H^i(\xi)$$

3

and this equations can be obtained by variation of the action

$$S = \int [\theta_k(\xi)\dot{\xi}^k - H(\xi)]dt \qquad (1.3)$$

If the Hamiltonian system is constructed from the non-singular Lagrangian [3], then the phase space $\mathcal{M}$ is a cotangent bundle over the configuration space of the corresponding Lagrangian system. In that case we have a separation of all coordinates $\xi^k, (k = 1, ..., 2N)$ into two canonically conjugated parts. The first part is formed by "coordinates" (q^α) of the configuration space and the second by "momenta" (p_α) $(\alpha = 1, ..., N)$. The latter are unbounded $(-\infty < p_\alpha < +\infty)$ and we can use the standard scheme of canonical quantization with the rule:

$$p_\alpha \longrightarrow \hat{p}_\alpha = -i\hbar \frac{\partial}{\partial q_\alpha} \qquad (1.4)$$

According to Darboux's theorem, the canonical coordinates exist on an arbitrary symplectic manifold; but in general, such coordinates exist only locally [1], and there is no global cotangent bundle structure with unbounded momenta. Consistent quantization requires a realization of not only the classical commutation relations, but of spectral conditions as well. Respectively, in general, the rule (1.4) is not acceptable, since the spectra of the differential operators are unbounded.

Note, that a symplectic manifold of general type naturally arises for the systems with singular Lagrangian (for example for gauge theories), when we apply the Dirac's procedure for constrained dynamics [3].

To generalize quantization method for such cases too we introduce some auxiliary Hamiltonian system with the phase space $T^*\mathcal{M}$, where $T^*\mathcal{M}$ is the cotangent bundle over the symplectic manifold $\mathcal{M}$. The new system has $4N$ dimension, and we choose the 1-form $\Theta = P_k d\xi^k$, where (P_k, ξ^k) are the standard coordinates on the cotangent bundle $T^*\mathcal{M}$: $P_k = P(\partial_{\xi^k})$. So, the coordinates P_k play the role of "momenta", while the ξ^k are "coordinates". The corresponding symplectic form is canonical: $d\Theta = dP_k \wedge d\xi^k$, and for the Poisson brackets of the new system we have (compare with (1.2'))

$$\{\xi^k, \xi^l\}_* = 0 = \{P_k, P_l\}_* \qquad \{P_k, \xi^l\}_* = \delta_k^l \qquad (1.5)$$

The index $*$ is used to make difference between the Poisson brackets (1.2) and (1.5). Below we denote the initial system by M, and the extended new system by T^*M.

Since the symplectic form ω is non-degenerated, the relation [1]

$$\omega(\Phi, \ \cdot \ ) = (\theta - P)\,(\cdot) \qquad (1.6)$$

defines the vector field Φ $(\Phi \in V(\mathcal{M}))$ uniquely. The components of this field Φ^k are given by

$$\Phi^k = \omega^{kl}(P_l - \theta_l) \qquad (1.6')$$

---

[1] $\omega(\Phi, \ \cdot \ )$ denotes the contraction of ω with Φ: $\omega(\Phi, \ \cdot \ )_l = \Phi^k \omega_{kl}$.

and respectively, we get the map $(T^*\mathcal{M} \mapsto V(\mathcal{M}))$ of the cotangent bundle $T^*\mathcal{M}$ to the space of vector fields on $\mathcal{M}$. Using this vector field Φ and some observable $f(\xi) \in \mathcal{O}(\mathcal{M})$ we can construct the function Φ_f on $T^*\mathcal{M}$

$$\Phi_f \equiv \Phi(f) = \Phi^k \partial_k f \tag{1.7}$$

and from $(1.6')$ we have

$$\Phi_f = \theta(V_f) - P(V_f) \tag{1.7'}$$

where V_f is the Hamiltonian vector field (1.1).

The definition of functions Φ_f by (1.7) at the same time gives the map

$$\mathcal{O}(\mathcal{M}) \mapsto \mathcal{O}(T^*\mathcal{M})$$

of observables of the system M to the certain class of functions on $T^*\mathcal{M}$. Then, from $(1.5 \text{ -} 1.7')$ we obtain

$$\{\Phi_f, \Phi_g\}_* = -\{f, g\} - \Phi_{\{f,g\}} \quad \{\Phi_f, g\}_* = -\{f, g\} \tag{1.8}$$

Note, that these commutation relations are written for the system T^*M, and here for the functions $\{f, g\}$ and g we use the same notations as for the corresponding observables on $\mathcal{M}$. Strictly, of course, we should distinguish between these functions. However, it is generally simpler not to do this except in case of possible confusion.

Now, let us introduce a new map from $\mathcal{O}(\mathcal{M})$ to $\mathcal{O}(T^*\mathcal{M})$

$$f \mapsto R_f \equiv f - \Phi_f \tag{1.9}$$

which in local coordinates (P_k, ξ^k) takes the form

$$R_f = f(\xi) + \partial_k f(\xi) \omega^{kl}(\xi)(P_l - \theta_l(\xi)) \tag{1.9'}$$

The 1-form θ in $(1.9')$ is assumed to be fixed, and the map (1.9) defines the class of observables R_f uniquely. We have that $R_f \neq R_g$ whenever $f \neq g$. Note that a change of the 1-form θ by an exact form dF: $\theta_k(\xi) \longrightarrow \theta_k(\xi) + \partial_k F(\xi)$, corresponds to

$$R_f \to R_f + \{f, F\} \tag{1.9''}$$

and for the system T^*M it is the canonical transformation generated by the function $F(\xi)$.

Then, using (1.8), for the Poisson brackets of constructed observables (1.9), we obtain

$$\{R_f, R_g\}_* = R_{\{f,g\}} \tag{1.10}$$

We choose the Hamiltonian of the extended system T^*M to be equal to R_H, where $H = H(\xi)$ is the initial Hamiltonian. Respectively, for the system T^*M the action (1.3) takes the form

$$S_{T^*\mathcal{M}} = \int [P_k(\xi)\dot{\xi}^k - R_H(P, \xi)]dt \tag{1.11}$$

5

The linear map (1.9) has two remarkable properties:

1. It preserves the Poisson brackets (see (1.10)).

2. The functions R_f in (1.9) contain the momentum variables P_k no higher than in the first degree.

Below we use these properties for the construction of the corresponding quantum operators.

As it was mentioned above, the system T^*M can be quantized by the scheme of canonical quantization. This means that the Hilbert space $\tilde{\mathcal{H}}$ is the space of squad integrable functions $\Psi(\xi)$ on $\mathcal{M}$: $\tilde{\mathcal{H}} = \mathcal{L}_2(\mathcal{M})$. It is convenient to introduce the invariant measure on $\mathcal{M}$

$$d\mu(\xi) \equiv \sqrt{\omega(\xi)}\, d^{2N}\xi \quad \text{with} \quad \omega(\xi) \equiv det\ \omega_{kl}(\xi) \tag{1.12}$$

and to define the scalar product by

$$\langle \Psi_2 | \Psi_1 \rangle = \int d\mu(\xi)\ \Psi_2^*(\xi)\Psi_1(\xi) \tag{1.13}$$

According to the scheme of canonical quantization for the function $f(\xi)$ we have the corresponding operator $\hat{f}$ which acts on a wave function $\Psi(\xi)$ as the multiplication by $f(\xi)$. Taking into account the remarks after the equations (1.8), we use the same notation $f(\xi)$ for this operator $\hat{f}$ as well: $\hat{f} \equiv f(\xi)$.

Further, the rule (1.4) defines the Hermitian operators $\hat{P}_k$

$$\hat{P}_k = -i\hbar\partial_k - i\hbar\frac{\partial_k\omega(\xi)}{4\omega(\xi)} \tag{1.14}$$

where the additional term, proportional to $\partial_k\omega$, arises from the measure (1.12) in (1.13).

Construction of Hermitian operators, in general, has an ambiguity connected to the ordering of coordinate and momentum operators in the functions of corresponding observables. For the functions R_f this ordering problem is only for the term $\partial_k f\omega^{kl}P_l$ (see (1.9')). When the momentum operator is only in the first degree, it is easy to see, that the following symmetric ordering

$$\partial_k f\omega^{kl}P_l \longrightarrow \frac{1}{2}(\partial_k f\omega^{kl}\hat{P}_l + \hat{P}_l\partial_k f\omega^{kl}) \tag{1.15}$$

defines a Hermitian operator, and for those operators there are no anomalies in the quantum commutation relations. Now, choosing the ordering (1.15) in (1.9'), and using that [2]

$$\partial_k(\sqrt{\omega}\ \omega^{kl}) = 0 \tag{1.16}$$

we obtain

$$\hat{R}_f = f(\xi) - \theta(V_f) - i\hbar V_f \tag{1.17}$$

___
[2]The formula (1.16) is a consequence of the Jacobi identity: $\omega^{il}\partial_l\omega^{jk} + \omega^{jl}\partial_l\omega^{ki} + \omega^{kl}\partial_l\omega^{ij} = 0$

where V_f is the Hamiltonian vector field (1.1), and $\theta(V_f)$ is the value of the 1-form θ on this field: $\theta(V_f) = \theta_k \omega^{kl} \partial_l f$. So, the operator $\hat{R}_f$ is constructed from the invariant terms, and it does not depend on the choice of coordinates ξ^k on $\mathcal{M}$.

Note, that a change of a 1-form θ by an exact form dF corresponds to the unitary transformation of operators $\hat{R}_f$ (see (1.9''))

$$\hat{R}_f \longrightarrow e^{-\frac{i}{\hbar}F(\xi)} \hat{R}_f e^{\frac{i}{\hbar}F(\xi)}$$

Since the operator ordering (1.15) avoids anomalies in the commutation relations, from (1.10) we get

$$[\hat{R}_f, \hat{R}_g] = -i\hbar \hat{R}_{\{f,g\}} \tag{1.18}$$

and this is the most interesting point of the described quantization scheme on the cotangent bundle of a symplectic manifold.

It is remarkable, that the operators (1.17) (which arise naturally in our scheme) are the pre-quantization operators of the theory of geometric quantization, and a representation of Poisson brackets algebra by these operators is a well known fact from this theory [1].

After canonical quantization on the cotangent bundle $T^*\mathcal{M}$ our goal is to use this quantum theory for the quantization of the initial system M, and in the next section we consider the connection between these two systems.

2 Constraints on a Cotangent Bundle

Geometrically there is a standard projection $(\pi : T^*\mathcal{M} \to \mathcal{M})$ of the cotangent bundle $T^*\mathcal{M}$ to the initial phase space $\mathcal{M}$. To find the dynamical relation between these two systems we introduce the constraint surface on the cotangent bundle $T^*\mathcal{M}$, and define it as the kernel of the mapping $T^*\mathcal{M} \to V(\mathcal{M})$ given by (1.6)-(1.6'). It means that on the constraint surface the vector field Φ vanishes: $\Phi = 0$, and if we use the functions $\Phi_f(P, \xi)$ (see (1.7)) this surface can be written as

$$\Phi_f = 0, \qquad \forall \ f(\xi) \in \mathcal{O}(\mathcal{M}) \tag{2.1}$$

From (1.8) and (1.9) we have

$$\{R_f, \Phi_g\}_* = \Phi_{\{f,g\}} \tag{2.2}$$

and we see, that (2.1), i.e. the constraint surface, is invariant under the canonical transformations generated by the functions R_f. In particular it is invariant in dynamics generated by the Hamiltonian R_H. Note, that the 1-form θ is assumed to be fixed in all these formulas.

In local coordinates the surface (2.1) can be written as

$$P_k - \theta_k(\xi) = 0 \tag{2.3}$$

7

(see $(1.6')$ -$(1.7')$), and respectively, the momenta P_k are defined uniquely. Hence, the coordinates $\xi^k (k = 1, ...2N)$ can be used for the parameterization of the constraint surface, and this surface is diffeomorphic to the manifold $\mathcal{M}$. Then, the reduction procedure gives (see (2.3) and $(1.9')$)

$$P_k d\xi^k|_{\Phi=0} = \theta_k(\xi)d\xi^k \qquad R_H|_{\Phi=0} = H(\xi)$$

and the action (1.11) of the system T^*M is reduced to (1.3). Thus, we conclude that the classical system T^*M on the constraint surface $\Phi_f = 0$ is equivalent to the initial one.

To find the connection on the quantum level too, we have to quantize the system T^*M taking into account the constraints (2.1).

Before beginning the quantum part of the reduction scheme, let us note, that the constraints (2.2) are written for an arbitrary observable $f(\xi)$, and since the constraint surface $\Phi = 0$ is $2N$ dimensional, only the finite number of those constraints are independent.

To select the independent constraints we introduce the complete set of observables on $\mathcal{M}$. The set of observables $\{f_n(\xi) \in \mathcal{O}(\mathcal{M}); (n = 1, ..., K)\}$ is called complete, if any observable $f(\xi) \in \mathcal{O}(\mathcal{M})$ can be expressed as a function of this set

$$f = \mathcal{F}(f_1, ..., f_K) \tag{2.4}$$

It is clear that $K \geq 2N$, and we can choose the set with $K = 2N$ only for the manifolds with global coordinates. For $K > 2N$ there are some functional relations for the set $f_1, ..., f_K$, and locally only 2N of these functions are independent. Then, from (1.7) and (2.4) we have

$$\Phi_f = \frac{\partial \mathcal{F}}{\partial f_n}\Phi_{f_n} \tag{2.5}$$

and the constraints (2.1) for arbitrary f are equivalent to K constraints

$$\Phi_{f_n} = 0, \quad (n = 1, ..., K) \tag{2.6}$$

In particular, in case of global coordinates we can introduce only $2N$ constraints $\Phi_{f_n}, (n = 1, ..., 2N)$. If it is not specified, below we are assuming that a manifold $\mathcal{M}$ has global coordinates and a set of functions $f_1, ..., f_{2N}$ is complete. Note, that the constraint surface and the reduced classical system are independent on the choice of such complete set. Using (1.8), we see that on the constraint surface (2.1) the rank of the matrix $\{\Phi_{f_n}, \Phi_{f_m}\}_*$ is equal to $2N$, and therefore, these constraints, in Dirac's classification, are the second class constraints.

For the constrained systems there are, actually, two schemes of quantization:
A. "First reduce and then quantize".
B. "First quantize and then reduce".

By the scheme A we are returning to the initial problem of quantization on the manifold $\mathcal{M}$. Therefore, it is natural to use the scheme B, especially as, the first step of this scheme we have already accomplished.

To justify our strategy it is necessary to show, that the schemes A and B give equivalent quantum theories, when the system M is quantizable by the canonical method, and also, it is worthwhile to have a certain general receipt for accounting the constraints (2.6) on the quantum level.

According to the scheme B the next step is a construction of constraint operators. From (1.9) and (1.17) the operators

$$\hat{\Phi}_f = i\hbar V_f + \theta(V_f) \tag{2.7}$$

are Hermitian, and by direct calculation one obtains

$$[\hat{\Phi}_f, \hat{\Phi}_g] = i\hbar(\{f, g\} + \hat{\Phi}_{\{f,g\}}) \tag{2.7'}$$

$$[\hat{R}_f, \hat{\Phi}_g] = -i\hbar \hat{\Phi}_{\{f,g\}} \tag{2.7''}$$

These commutators are quantum versions of the relations (1.8) and (2.2). As it was expected, there are no anomalies for them (see (1.18)).

Now, we should make reduction of Hilbert space using the constraint operators (2.7) for some complete set of functions $f_1, ..., f_{2N}$. The reduced Hilbert space for the constrained systems is called the physical Hilbert space as well, and we denote it by $\mathcal{H}_{ph}$.

For systems with the second class constraints there is the following reduction procedure [3]: one has to select a commuting subset of N constraints [3] $\hat{\Phi}_1, ..., \hat{\Phi}_N$

$$[\hat{\Phi}_a, \hat{\Phi}_b] = 0 \qquad 1 \le a, b \le N$$

and then, construct a physical Hilbert space $\mathcal{H}_{ph}$ from the states which satisfy the Dirac's conditions $\hat{\Phi}_a|\Psi_{ph}\rangle = 0$, $a = 1, ..., N$. Note, that we can not put all constraints equal to zero in strong sense ($\hat{\Phi}_k|\Psi\rangle = 0$, $k = 1, ..., 2N$), since it contradicts to commutation relations of the second class constraints.

From (2.7') we see that in our case, the described procedure implies selection of N commuting observables f_a, $a = 1, ..., N$; $\{f_a, f_b\} = 0$, and further, solution of the differential equations

$$\hat{\Phi}_{f_a} \Psi_{ph}(\xi) = 0, \qquad a = 1, ..., N \tag{2.8}$$

Construction of physical states by selection of N commuting observables is quite natural from the point of view of standard quantum mechanics, and we shall return to this point later.

Equations (2.8) are the first order linear differential equations and, in principle, they can be explicitly integrated to describe corresponding wave functions. But at this stage of quantization scheme B two significant problems usually arise: the first is connected with the introduction of scalar product for the physical vectors [5], and the second, with the definition of observable operators on these vectors.

---

[3]If subset of constraints is treated as the first class (independently from others), then, in our case, they are commuting (see (2.7')).

For the first problem, the point is, that solutions of Dirac's conditions $\hat{\Phi}_a|\Psi_{ph}\rangle = 0$, in general, are not the vectors of the same Hilbert space where the first stage of quantization was accomplished (in our case $\mathcal{L}_2(\mathcal{M})$), and it is necessary to introduce the structure of Hilbert space additionally. These solutions, as a rule, are in the space dual to the Hilbert space, and one has to introduce the new scalar product for them.

In our case, solutions of (2.8), in general, are not square integrable on $\mathcal{M}$ (usually they are generalized functions), and the scalar product (1.13) needs modification. On the other hand, a certain measure in scalar product defines the class of functions square integrable by this measure. Thus, a measure for the new scalar product and the class of solutions of (2.8) should be adjusted.

One method for the solution of this problem is based on the introduction of complex constraints [6]. Note, that classical observables $f(\xi)$ are assumed to be real functions on a phase space, but it is clear, that the whole considered construction (except for the self-adjointness) can be naturally extended for complex valued functions $f(\xi) = f_1(\xi) + i f_2(\xi)$ as well.

Using the remaining part of constraints $\Phi_{f_{N+1}}, ..., \Phi_{f_{2N}}$, one can introduce constraints for the complex functions $Z_a = f_a + i\epsilon f_{N+a}$ and consider the equations

$$(\hat{\Phi}_{f_a} + i\epsilon\hat{\Phi}_{f_{N+a}})|\Psi_\epsilon\rangle = 0 \qquad a = 1, ..., N \qquad (2.9)$$

Here, $1 \leq a \leq N$, $\{f_a, f_{N+a}\} \neq 0$ and ϵ is some real parameter [4].

The condition (2.9) looks like Gupta-Bleuler quantization [7], and for normalizable solutions $|\Psi_\epsilon\rangle$ the mean values of corresponding constraints vanish

$$\langle\Psi_\epsilon|\hat{\Phi}_{f_a}|\Psi_\epsilon\rangle = 0 \qquad\qquad \langle\Psi_\epsilon|\hat{\Phi}_{f_{N+a}}|\Psi_\epsilon\rangle = 0 \qquad (2.10)$$

It turns out that the solutions of (2.9) could be square integrable indeed, and then, they form some subspace of the Hilbert space $\mathcal{L}_2(\mathcal{M})$ (see the below). The corresponding reduced physical Hilbert space we denote by $\mathcal{H}_\epsilon$. We have $\Psi_\epsilon(\xi) \in \mathcal{H}_\epsilon \subset \mathcal{L}_2(\mathcal{M}) \subset \mathcal{L}_2^*(\mathcal{M})$, where $\mathcal{L}_2^*(\mathcal{M})$ is the space dual to the Hilbert space $\mathcal{L}_2(\mathcal{M})$. If we consider the physical states $|\Psi_\epsilon\rangle$ as the vectors of the dual space $\mathcal{L}_2^*(\mathcal{M})$, then the suitable choice of the norms $||\Psi_\epsilon||$, and some smooth dependence on the parameter ϵ can provide existence of the limit

$$\lim_{\epsilon\to 0}|\Psi_\epsilon\rangle = |\Psi_{ph}\rangle$$

where $|\Psi_{ph}\rangle \in \mathcal{H}_{ph} \subset \mathcal{L}_2^*(\mathcal{M})$ (see Appendix A). Obtained physical states $|\Psi_{ph}\rangle$ specify the class of solutions of (2.8), and the scalar product for them is defined by

$$\langle\Psi_{2ph}|\Psi_{1ph}\rangle = \lim_{\epsilon\to 0}\frac{\langle\Psi_{2\epsilon}|\Psi_{1\epsilon}\rangle}{||\Psi_{2\epsilon}||\,||\Psi_{1\epsilon}||} \qquad (2.11)$$

where $|\Psi_{1ph}\rangle$ and $|\Psi_{2ph}\rangle$ are the limits of $|\Psi_{1\epsilon}\rangle$ and $|\Psi_{2\epsilon}\rangle$ respectively. Note, that in the limit $\epsilon \to 0$ the norm of vectors $||\Psi_\epsilon||$ usually diverges, but the scalar product

---

[4]Sometimes we omit the index "ph" for the physical vectors (and physical Hilbert space), and use the index ϵ only

(2.11) remains finite ($|\langle\Psi_{2ph}|\Psi_{1ph}\rangle| \le 1$). (for more details see Appendix A and the examples in the next section).

It is remarkable that the choice of physical states by the conditions (2.8) and (2.9) is equivalent to the choice of real and complex polarizations of geometric quantization respectively [1].

The second above mentioned problem is connected with the fact, that a reduced Hilbert space constructed by (2.8) (or (2.9)), in general, is not invariant under the action of some pre-quantization operator $\hat{R}_g$. Indeed, the invariance conditions for (2.8) are

$$[\hat{R}_g, \hat{\Phi}_{f_a}] = \sum_{b=1}^{N} d_a^b \hat{\Phi}_{f_b} \qquad (1 \le a \le N) \tag{2.12}$$

and, from (2.7″) we see that it is not valid for arbitrary $g(\xi)$. Moreover, even if a pre-quantization operator acts invariantly on $\mathcal{H}_{ph}$, this operator can be non-Hermitian on $\mathcal{H}_{ph}$, when the latter is not a subspace of $\mathcal{L}_2(\mathcal{M})$ and the Hilbert structure is introduced additionally (see the example below).

For the definition of the corresponding observable operator on the physical Hilbert space one can deform the pre-quantization operator adding quadratic (and higher) powers of constraint operators [5]. Then, using commutation relations (2.7′) and (2.7″), one can construct a new Hermitian operator, which is invariant on the reduced Hilbert space. Of course, there are different possible deformations, and in general, they define different operators on the physical Hilbert space. In terms of usual canonical quantization, different deformations correspond to different operator orderings. This is the standard ambiguity of quantum theories which in the classical limit $\hbar \to 0$ vanishes. Note, that corresponding deformed classical functions are indistinguishable on the constraint surface $\Phi_f = 0$.

The described quantization scheme we call E-quantization scheme. In the next section we consider application of this scheme for some simple examples. We use these examples as a test for our approach as well.

3 Examples of E-Quantization scheme

Example 1. Phase space is a plane $\mathcal{M} \equiv \mathcal{R}^2$ with standard coordinates $\xi^1 \equiv p$, $\xi^2 \equiv q$ and the symplectic form $\omega = dp \wedge dq$. The coordinates p and q are global and from (1.7′) we get

$$\Phi_p = \frac{1}{2}p - P_q \qquad \Phi_q = \frac{1}{2}q + P_p \tag{3.1}$$

where, for the convenience, we choose the 1-form $\theta = \frac{1}{2}pdq - \frac{1}{2}qdp$. The corresponding constraint operators are

$$\hat{\Phi}_p = \frac{1}{2}p + i\hbar\partial_q \qquad \hat{\Phi}_q = \frac{1}{2}q - i\hbar\partial_p \tag{3.2}$$

---

[5]Corresponding procedure in classical case is given in Appendix B

and, according to (2.9), for the physical vectors $|\Psi_\epsilon\rangle$ we have the equation

$$\left(\hat{\Phi}_q - i\epsilon\hat{\Phi}_p\right)|\Psi_\epsilon\rangle = 0 \tag{3.3}$$

with some positive parameter ϵ ($\epsilon > 0$) [6]. Solutions of (3.3) have the form

$$\Psi_\epsilon(p,q) = \exp\left(-\frac{\epsilon p^2}{2\hbar}\right)\exp\left(-\frac{ipq}{2\hbar}\right)\psi(q - i\epsilon p) \tag{3.4}$$

where ψ is an arbitrary function. For the square integrability of these solutions we can specify the class of ψ functions, for example, by

$$\psi(\xi) = \exp\left(-\frac{\gamma\xi^2}{2}\right)P(\xi) \qquad (\xi \equiv q - i\epsilon p) \tag{3.5}$$

Here γ is some fixed positive parameter ($\gamma > 0$), and $P(\xi)$ — any polynomial. Then, for sufficiently small ϵ the functions (3.4) will be square integrable on the plane and they form the physical subspace $\mathcal{H}_\epsilon$, ($\mathcal{H}_\epsilon \in \mathcal{L}_2(\mathcal{R}^2)$).

To investigate the case $\epsilon = 0$, we consider the limit $\epsilon \to 0$ of functions (3.4) (see Appendix A), and get

$$\Psi_{ph}(p,q) = \exp\left(-\frac{ipq}{2\hbar}\right)\psi(q) \tag{3.6}$$

It is clear, that these functions are not squared integrable on the plane, but they are well defined elements of the dual space $\Psi_{ph}(p,q) \in \mathcal{L}_2^*(\mathcal{R}^2)$. The functions (3.6) form the physical Hilbert space $\mathcal{H}_{ph}$, and they are solutions of (3.3) with $\epsilon = 0$. Using rule (2.11), we obtain

$$\langle\Psi_{2ph}|\Psi_{1ph}\rangle = \frac{1}{N_1 N_2}\int \psi_2^*(q)\psi_1(q)\,dq \tag{3.7}$$

where

$$N_i^2 = \int |\psi_i(q)|^2\,dq \qquad (i = 1,2; \quad N_i > 0)$$

Action of pre-quantization operators

$$\hat{R}_p = \frac{1}{2}p - i\hbar\partial_q \qquad \hat{R}_q = \frac{1}{2}q + i\hbar\partial_p \tag{3.8}$$

on the physical states (3.6) gives

$$\hat{R}_p\Psi_\epsilon(p,q) = \exp\left(-\frac{ipq}{2\hbar}\right)(-i\hbar)\psi'(q) \qquad \hat{R}_q\Psi_\epsilon(p,q) = \exp\left(-\frac{ipq}{2\hbar}\right)q\psi(q) \tag{3.9}$$

Thus, from (3.7) and (3.9) we have the standard coordinate representation of quantum mechanics. Similarly, one can obtain the momentum representation in the limit $\epsilon \to \infty$ with corresponding choice of the class of solutions (3.5).

Let us consider the problem of construction of some observable operators on the physical Hilbert space $\mathcal{H}_{ph}$ (3.6). It is easy to check that this space is invariant under

---

[6]For $\epsilon \leq 0$ equation (3.3) has no normalizable solutions.

the action of pre-quantization operators $\hat{R}_f$, where $f(p,q) = pA(q) + U(q)$, with arbitrary $A(q)$ and $U(q)$. But it turns out, that these operators $\hat{R}_f$ are Hermitian (with respect to the scalar product (3.7)) only for the constant function $A(q)$ ($A(q) = c$). Similarly, there is a problem of definition of kinetic energy operator, since the corresponding pre-quantization operator is not defined on the chosen $\mathcal{H}_{ph}$ [7]. These are just the problems mentioned at the end of the previous section, and for the definition of corresponding observable operators we can make appropriate deformations (see Appendix B). For example, deformation of the pre-quantization operator of kinetic energy $E = p^2/2m$ by the quadratic term

$$\hat{R}_{p^2/2m} \rightarrow \hat{R}_{p^2/2m} + \frac{1}{2m}\hat{\Phi}_p^2 \equiv \hat{E}$$

gives that the corresponding operator $\hat{E}$ is well defined on $\mathcal{H}_{ph}$, and effectively it acts as the standard kinetic energy operator

$$\hat{E} : \psi(q) \mapsto -\frac{\hbar^2}{2m}\psi''(q)$$

Now, we return to the physical subspace $\mathcal{H}_\epsilon$ with some fixed positive ϵ. In complex coordinates

$$z = \frac{q + i\epsilon p}{\sqrt{2\epsilon\hbar}} \qquad z^* = \frac{q - i\epsilon p}{\sqrt{2\epsilon\hbar}} \tag{3.10}$$

(3.3) takes the form

$$\left(\partial_z + \frac{z^*}{2}\right)\Psi_\epsilon(z, z^*) = 0 \tag{3.10'}$$

and the solutions are

$$\Psi_\epsilon(z, z^*) = \exp\left(-\frac{1}{2}|z|^2\right)F(z^*) \tag{3.11}$$

where $F(z^*)$ is any holomorphic function of z^*. Comparing (3.11) and (3.4) we have $F(z^*) = \exp\left(1/2\ z^{*2}\right)\psi(\sqrt{2\epsilon\hbar}\ z^*)$. From the point of view of canonical quantization the complex coordinates z and z^* (see (3.10)) are the classical functions of annihilation and creation operators $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^*$ respectively. The corresponding pre-quantization operators

$$\hat{R}_z = \frac{z}{2} + \partial_z^* \qquad\qquad \hat{R}_{z^*} = \frac{z^*}{2} - \partial_z$$

act invariantly on the physical Hilbert space $\mathcal{H}_\epsilon$, and we have

$$\hat{R}_z\Psi_\epsilon(z, z^*) = \exp\left(-\frac{1}{2}|z|^2\right)F'(z^*) \qquad\qquad \hat{R}_{z*}\Psi_{ph}(z, z^*) = \exp\left(-\frac{1}{2}|z|^2\right)z^*F(z^*)$$

Thus, the reduction on $\mathcal{H}_\epsilon$ gives the holomorphic representation of quantum mechanics [8], and we see that for the Example 1 the quantum theory of the E-quantization scheme is equivalent to the ordinary canonical one. For different ϵ the physical Hilbert spaces $\mathcal{H}_\epsilon$ are different subspaces of $\mathcal{L}_2(\mathcal{R}^2)$, and the corresponding representations of

---

[7]For this $\mathcal{H}_{ph}$, such problem have functions containing momentum p in second and higher degrees

13

canonical commutation relations are unitary equivalent due to Stone - von-Neumann theorem [9].

Further, for any state $|\Psi\rangle$ of standard quantum mechanics we have

$$\langle p, q; \epsilon|\Psi\rangle = \int dx \ \langle p, q; \epsilon|x\rangle\psi(x) \tag{3.12}$$

where $\psi(x) \equiv \langle x|\Psi\rangle$ is a wave function of coordinate representation, $|p, q; \epsilon\rangle$ is a coherent state [10]

$$\mathbf{\hat{a}}|p, q; \epsilon\rangle = \frac{q + i\epsilon p}{\sqrt{2\epsilon\hbar}} \ |p, q; \epsilon\rangle \tag{3.13}$$

and respectively, the "matrix element" $\langle p, q; \epsilon|x\rangle$ is given by

$$\langle p, q; \epsilon|x\rangle = \left(\frac{1}{\pi\epsilon\hbar}\right)^{1/4} \exp\left(\frac{i}{2\hbar}pq\right) \exp\left(-\frac{i}{\hbar}px\right) \exp\left(-\frac{(x - q)^2}{2\epsilon\hbar}\right) \tag{3.13$'$}$$

Then, from (3.12) and (3.13$'$) we obtain

$$\lim_{\epsilon \to 0} \ \langle p, q; \epsilon|\Psi\rangle \left(\frac{1}{4\pi\epsilon\hbar}\right)^{1/4} = \exp\left(-\frac{ipq}{2\hbar}\right) \psi(q) \tag{3.14}$$

It is well known that the matrix element $\langle p, q; \epsilon|\Psi\rangle$ defines the wave function of holomorphic representation (see [8, 10])

$$\langle p, q; \epsilon|\Psi\rangle = \exp\left(-\frac{1}{2}|z|^2\right)F(z^*) \equiv \tilde{\Psi}_\epsilon(p, q) \tag{3.15}$$

where the variables p, q and z, z^* are related by (3.10). On the other hand, from the equivalence of holomorphic representation and E-quantization scheme, the wave function $\tilde{\Psi}_\epsilon(p, q)$ in (3.15) can be considered as the vector of physical Hilbert space $\mathcal{H}_\epsilon$ (compare (3.11) and (3.15)). Then, (3.12) and (3.14) will be similar to (3.4) and (3.6) respectively. Only, it should be noted, that the two physical states $\Psi_\epsilon(p, q)$ and $\tilde{\Psi}_\epsilon(p, q)$, constructed by the same function $\psi(q) \in \mathcal{L}_2(\mathcal{R}^1)$, are different ($\Psi_\epsilon(p, q) \neq \tilde{\Psi}_\epsilon(p, q)$) (see (3.4) and (3.12)), and they coincide only in the limit $\epsilon \to 0$. This short remark indicates different possibilities of described limiting procedure (for more details see Appendix A).

Example 2. Phase space is a cylinder $\mathcal{M} \equiv \mathcal{R}^1 \times \mathcal{S}^1$ with the coordinates $\xi^1 \equiv S \in \mathcal{R}^1$, $\xi^2 \equiv \varphi \in S^1$ and the symplectic form $\omega = dS \wedge d\varphi$. This is a model of rotator where S is an angular momentum.

Since a cylinder is a cotangent bundle over a circle, the canonical quantization for this model is realized on the space of square integrable functions $\psi(\varphi)$ on a circle ($\psi(\varphi) \in \mathcal{L}_2(\mathcal{S}^1)$). The quantization rule (1.4) gives

$$\hat{S} \ \psi(\varphi) = -i\hbar\partial_\varphi\psi(\varphi) \quad \hat{\cos}\varphi \ \psi(\varphi) = \cos\varphi \ \psi(\varphi) \quad \hat{\sin}\varphi \ \psi(\varphi) = \sin\varphi \ \psi(\varphi) \tag{3.16}$$

and the operator $\hat{S}$ has the discrete spectrum $S_n = n\hbar$, ($n \in Z$), with the eigenfunctions $\psi_n(\varphi) = 1/\sqrt{2\pi} \ \exp\left(in\varphi\right)$.

14

The coordinate φ is not global, and for the 1-form we choose $\theta = S d\varphi$. The set of functions

$$f_1 = S, \quad f_2 = \cos\varphi, \quad f_3 = \sin\varphi \tag{3.17}$$

is complete (with the relation $f_2^2 + f_3^2 = 1$), and for the corresponding constraint operators we get

$$\hat{\Phi}_S = S + i\hbar\partial_\varphi, \quad \hat{\Phi}_{\cos\varphi} = i\hbar\sin\varphi\partial_S, \quad \hat{\Phi}_{\sin\varphi} = -i\hbar\cos\varphi\partial_S \tag{3.17'}$$

Note, that there is a possibility to have the complete set with only two functions as well. For example,

$$\tilde{f}_1 = e^{S/\lambda}\cos\varphi \qquad \tilde{f}_2 = e^{S/\lambda}\sin\varphi \tag{3.18}$$

where λ is some constant parameter (with dimension of angular momentum). These functions are global coordinates on a cylinder and they give the map of a cylinder on to a plane without origin: $(\tilde{f}_1, \tilde{f}_2) \in \mathcal{R}^2 - \{0\}$.

From (3.17′) we see that in the E-quantization scheme the wave functions $\psi(\varphi)$ of "φ representation" can be obtained by

$$\hat{\Phi}_{\cos\varphi}\,\Psi_{ph}(S,\varphi) = 0 \quad \text{and} \quad \hat{\Phi}_{\sin\varphi}\,\Psi_{ph}(S,\varphi) = 0 \tag{3.19}$$

But it is clear that these functions are not normalizable on the cylinder. Situation with the condition

$$\hat{\Phi}_S\,\Psi_{ph}(S,\varphi) = 0 \tag{3.20}$$

is more complicated, since equation (3.20) has no global regular solutions. In the class of generalized functions one can find the solutions of the type

$$\Psi_{ph,n} = \delta(S - n\hbar)\exp(in\varphi) \quad (n \in Z) \tag{3.21}$$

which obviously are not square integrable on the cylinder. To investigate these classes we need a limiting procedure as it was done for Example 1. Such a procedure we consider in the next section with some motivation and generalization of condition (2.9), and here, in the rest part of this section, we construct some physical Hilbert spaces as the subspaces of $\mathcal{L}_2(\mathcal{R}^1 \times \mathcal{S}^1)$. For this we introduce the complex coordinates related to (3.18)

$$z = \tilde{f}_1 - i\tilde{f}_2 = \exp(S/\lambda - i\varphi) \qquad z^* = \tilde{f}_1 + i\tilde{f}_2 = \exp(S/\lambda + i\varphi) \tag{3.22}$$

and impose condition like (2.9) for $\epsilon = 1$

$$\hat{\Phi}_{z^*}\,|\Psi_{ph}\rangle = 0 \tag{3.23}$$

This is equivalent to the equation

$$\left(\partial_z + \frac{\lambda}{2\hbar}\frac{\log|z|}{z}\right)\Psi_{ph}(z,z^*) = 0 \tag{3.24}$$

15

and for the solutions we get

$$\Psi_{ph}(z, z^*) = \exp\left(-\frac{\lambda}{2\hbar}(\log|z|)^2\right)\psi(z^*) \qquad (3.25)$$

where $\psi(z^*)$ is any holomorphic function $(\partial_z\psi = 0)$ on the plane without origin and it has the expansion

$$\psi(z^*) = \sum_{n=-\infty}^{\infty} d_n z^{*n}$$

Respectively, in (S, φ) coordinates (3.25) takes the form

$$\Psi_{ph}(S, \varphi) = \sum_{n=-\infty}^{\infty} c_n \exp\left(-\frac{(S - n\hbar)^2}{2\lambda\hbar}\right)\exp(in\varphi) \qquad (3.26)$$

with $c_n = d_n \exp(\hbar n^2/2\lambda)$, and square integrability gives

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty \qquad (3.27)$$

From (1.9) and (3.17$'$), the pre-quantization operator of angular momentum is $\hat{R}_S = -i\hbar\partial_\varphi$. It is a well defined operator on the physical subspace (3.26), and has the same non-degenerated spectrum, as the operator $\hat{S}$ of the canonical quantization. Thus, we see the unitary equivalence of these two quantizations.

4 Minimal Fluctuations of Quantum Constraints

For Example 1 of the previous section the constraint operators $\hat{\Phi}_p$ and $\hat{\Phi}_q$ have the canonical commutation relations (see (3.2))

$$[\hat{\Phi}_p, \hat{\Phi}_q] = i\hbar \qquad (4.1)$$

The condition (3.3) is equivalent to the choice of physical states $|\Psi_\epsilon\rangle$ as the "vacuum" states in Φ_p, Φ_q variables [8]. Then, the mean values of constraints are equal to zero

$$\langle\Psi_\epsilon|\hat{\Phi}_p|\Psi_\epsilon\rangle = 0 \qquad\qquad \langle\Psi_\epsilon|\hat{\Phi}_q|\Psi_\epsilon\rangle = 0 \qquad (4.1')$$

and the product of quadratic fluctuations is minimal

$$\langle\Psi_\epsilon|\hat{\Phi}_p^2|\Psi_\epsilon\rangle\langle\Psi_\epsilon|\hat{\Phi}_q^2|\Psi_\epsilon\rangle = \frac{\hbar^2}{4} \qquad (4.1'')$$

---

[8]Recall that due to quantum uncertainties, we can not put $\hat{\Phi}_p|\Psi\rangle = 0$ and $\hat{\Phi}_q|\Psi\rangle = 0$ simultaneously.

Thus, the meaning of the condition (2.9) for this simple example is that the obtained physical states $|\Psi_\epsilon\rangle$ provide the best realization of the classical constraints $\Phi_p = 0$, $\Phi_q = 0$ on the quantum level.

Let us consider the condition (2.9) in general case. Note, that if two functions f_a and f_{N+a} are canonically conjugated: $\{f_a, f_{N+a}\} = 1$, then the corresponding constraint operators have canonical commutation relations (see (2.7′)). Therefore, for the construction of physical states by (2.9) it is natural to choose the function f_{N+a} as a canonically conjugated to f_a, and repeat calculations of Example 1 in f_a, f_{N+a} variables. Unfortunately, this simple procedure, in general, fails. The reason is that the canonically conjugated variable f_{N+a} usually exists only locally and corresponding constraint $\Phi_{f_{N+a}}$ is not well defined both on classical and quantum levels. For example, canonically conjugated variable to the harmonic oscillator Hamiltonian $H = 1/2(p^2 + q^2)$ is the polar angle α

$$p = \sqrt{2H} \cos \alpha \qquad q = \sqrt{2H} \sin \alpha \qquad (4.2)$$

Choosing the 1-form $\theta = 1/2(pdq - qdp)$, we get

$$\hat{\Phi}_H = H + i\hbar \partial_\alpha \qquad (4.2′)$$

and for the operator $\hat{\Phi}_\alpha$ one can formally write $\hat{\Phi}_\alpha = -i\hbar \partial_H$, but this operator is not self-adjoint . Then, though the equation

$$(\hat{\Phi}_H + i\epsilon \hat{\Phi}_\alpha)|\Psi\rangle = 0 \qquad (4.2″)$$

has integrable solutions (for example $\Psi(p,q) = \exp(-H^2/2\epsilon\hbar)$), nevertheless they are not acceptable for the physical states, since the mean values of the constraint operators $\hat{\Phi}_H$ and Φ_α do not vanish, and minimization of quadratic fluctuations is not achieved as well.

For $\epsilon = 0$ one can write the formal solution of (4.2″) (like (3.21)): $\Psi = \delta(H - \hbar n) \exp(in\alpha)$, and since $H \geq 0$, such "solutions" are non-zero only for $n \geq 0$. Then, the pre-quantization operator $\hat{R}_H = -i\hbar \partial_\alpha$ has the spectrum $H_n = \hbar n, n \geq 0$. The situation is similar for any completely integrable system [11]. In action-angle variables I_a, φ_a $(a = 1, ..., N)$ we have the 1-form $\theta = I_a d\varphi_a$ and the Hamiltonian $H = H(I_1, ..., I_N)$. Then, the constraint and pre-quantization operators take the form

$$\hat{\Phi}_{I_a} = I_a + i\hbar \partial_{\varphi_a} \qquad (4.3)$$

$$\hat{R}_{I_a} = -i\hbar \partial_{\varphi_a} \qquad \hat{R}_H = H - \frac{\partial H}{\partial I_a} \hat{\Phi}_{I_a} \qquad (4.3′)$$

If φ_a are the cyclic variables $(\varphi_a \in \mathcal{S}^1)$ [9], then by described formal operations we obtain the "physical states"

$$\Psi_{ph}(I, \varphi) = \prod_{a=1}^{N} \delta(I_a - \hbar n_a) \exp(in_a \varphi_a) \qquad (4.4)$$

---

[9]Note, that operator $\hat{\Phi}_{\varphi_a}$ is Hermitian, when the corresponding action variable is unbounded $-\infty \leq I_a \leq \infty$ (as the angular momentum S for the Example 2).

as the "solutions" of the equations

$$\hat{\Phi}_{I_a}\Psi_{ph}(I,\varphi) = 0 \qquad\qquad (4.4')$$

The spectra of pre-quantization operators $(4.3')$ on these "physical states" are

$$(I_a)_{n_a} = \hbar n_a \qquad \text{and} \qquad H_{n_1,\dots,n_N} = H(\hbar n_1, \dots, \hbar n_N)$$

where n_a are integer numbers, and corresponding admissible values are chosen according to the possible classical values of the variables I_a (as, for example, $n \geq 0$ for the harmonic oscillator). It is remarkable, that these formal results correspond to the quantization rule

$$I_a\Delta\varphi_a = \oint p_a dq_a = 2\pi\hbar n_a \qquad\qquad (4.5)$$

which is almost the semi-classical one. From these formal operations it seems that the quantum problem is solvable for any completely integrable system; but of course, all these expressions here have only symbolic meaning and (4.4) needs further specification, taking account of N other constraints and limiting procedure as well.

After these remarks let us consider the case when observables f_a and f_{N+a} (in (2.9)) are not canonically conjugated to each other. For the convenience we use the notations $f_a \equiv f$, $f_{N+a} \equiv g$ and introduce corresponding constraint operators $\hat{\Phi}_f$ and $\hat{\Phi}_g$.

It turns out, that in general, equation (2.9) has no normalizable solutions at all, and choice of sign (or value) of ϵ does not help[10]. For example, if f is a kinetic energy $f = p^2/2m$, and g is a coordinate $g = q$ of one dimensional system, then (2.9) takes the form (with $\theta = pdq$ and $m = 1$)

$$(p^2 + i\hbar p\partial_q + \epsilon\hbar\partial_p)\Psi_\epsilon(p,q) = 0$$

and the solutions

$$\Psi_\epsilon(p,q) = \exp\left(-\frac{\epsilon p^3}{3\epsilon\hbar}\right)\psi(p^2 + 2i\epsilon q)$$

evidently are not normalizable. Of course, for this example we can return to the canonical coordinates p, q and make reduction (3.3) with constraints Φ_p and Φ_q; but if we intend to deal with arbitrary observables and symplectic manifolds, we have to generalize the condition (2.9). For this we introduce the minimization principle for quadratic fluctuations.

Quadratic fluctuations of two Hermitian operators $\hat{\Phi}_f$ and $\hat{\Phi}_g$ can be characterized by the functional $U(\Psi)$

$$U(\Psi) \equiv \langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle \qquad\qquad (4.6)$$

where $|\Psi\rangle$ is a vector with the unit norm $\langle\Psi|\Psi\rangle = 1$.

Then, one can postulate the principle that the physical states provide minimization of this functional (see $(4.11')$). For two arbitrary Hermitian operators minimization

---

[10]Sometimes, even normalizable solutions are not acceptable as well (see (4.2) and Section 5)

problem of uncertainties was studied in [12], and in Appendix C we present some results of this investigation. Note, that in [12] the minimization problem was considered for another functional $U_1(\Psi)$

$$U_1(\Psi) \equiv \frac{\langle \Psi | \hat{\Phi}_f^2 | \Psi \rangle \langle \Psi | \hat{\Phi}_g^2 | \Psi \rangle}{\langle \Psi | \hat{A} | \Psi \rangle^2} \tag{4.7}$$

as well. Here the operator $\hat{A}$ is the commutator

$$\hat{A} = -\frac{i}{\hbar}[\hat{\Phi}_f, \hat{\Phi}_g] \tag{4.8}$$

and only for the c-number operator the functionals $U(\Psi)$ and $U_1(\Psi)$ are equivalent. In this section we consider only the functional $U(\Psi)$.

Then, using results of [12] (see (C.4) and (C.5)), minimization principle gives that the physical wave functions $|\Psi_{ph}\rangle$ can be obtained from the equation

$$\frac{1}{2a^2}\hat{\Phi}_f^2|\Psi_{ph}\rangle + \frac{1}{2b^2}\hat{\Phi}_g^2|\Psi_{ph}\rangle = |\Psi_{ph}\rangle \tag{4.9}$$

and subsidiary conditions

$$a^2 = \langle \Psi_{ph}|\hat{\Phi}_f^2|\Psi_{ph}\rangle \qquad b^2 = \langle \Psi_{ph}|\hat{\Phi}_g^2|\Psi_{ph}\rangle \tag{4.10}$$

where a and b are some fixed parameters. Possible values of these parameters are defined from the following procedure: At first we have to solve the equation (4.9) with free parameters a, b and select the solutions with unit norm which satisfy (4.10). Usually after this we still have a freedom in a and b. Then we must choose one of those pairs with minimal product of ab (we assume both a and b to be nonnegative). The fixed values of the parameters a and b provide that the solutions of (4.9) form the linear space as the subspace of $\mathcal{L}_2(\mathcal{M})$. This subspace should define the physical Hilbert space $\mathcal{H}_{ph} \equiv \mathcal{H}_{(a,b)}$ of the system.

Thus, instead of the first order differential equation (2.9) with one parameter ϵ (see (2.9)) we get the second order equation (4.9) with two parameters a, b and subsidiary conditions (4.10). Note , that possible limiting procedure in (4.9) for $a \to 0$ (or $b \to 0$) can specify the physical states $|\Psi_{ph}\rangle$ with $\hat{\Phi}_f|\Psi_{ph}\rangle = 0$ (or $\hat{\Phi}_g|\Psi_{ph}\rangle = 0$).

For the test of formulated principle, at first we consider again Example 1. In this case the constraint operators $\hat{\Phi}_f \equiv \hat{\Phi}_p$ and $\hat{\Phi}_g \equiv \hat{\Phi}_q$ have the canonical commutation relations (4.1′). Then, (4.9) looks like the harmonic oscillator eigenvalue problem with the frequency $\omega = 1/ab$ and the eigenvalue $E = 1$. Respectively, we get $\hbar(n + 1/2) = ab$. One can check, that all the oscillator's eigenstates $|n\rangle$ satisfy the conditions (4.10), and therefore the minimal ab ($ab = \hbar/2$) corresponds to the vacuum state ($n = 0$) given by $(a\hat{\Phi}_q - ib\hat{\Phi}_p)|\Psi_{ph}\rangle = 0$. Thus, for the physical states we arrive again to (3.3) with $\epsilon = b/a$, and the limiting procedure $a \to 0$ (or $b \to 0$) can can be accomplished in a similar way.

Now, let us consider Example 2 with the constraint operators (3.17′). For the convenience we can construct the operator $\hat{O} \equiv \hat{\Phi}_{\sin\varphi}^2 + \hat{\Phi}_{\cos\varphi}^2$, and minimize the

19

product $\langle\Psi|\hat{\Phi}_S^2|\Psi\rangle\langle\Psi|\hat{O}|\Psi\rangle$. From (3.17′) we have $\hat{O} = -\hbar^2\partial_S^2$, and we see that this operator is a square of the Hermitian operator $\hat{\Phi}_\varphi \equiv -i\hbar\partial_S$ ($\hat{O} = \hat{\Phi}_\varphi^2$). Then, from the variation principle we get the equation (4.9) with $\hat{\Phi}_f = S + i\hbar\partial_\varphi$ and $\hat{\Phi}_g = -i\hbar\partial_S$. Since these two Hermitian operators have canonical commutation relations, we arrive again at the oscillator problem. Only, now the "ground" state should be obtained from the equation

$$(S + i\hbar\partial_\varphi + \frac{a}{b}\hbar\partial_S)|\Psi_{ph}\rangle \tag{4.11}$$

Hence, for this example, using the minimization principle, we arrive at the equation (4.11). It is interesting to note, that the equations (4.11) and (3.24) are equivalent, and the functions (3.26) with $\lambda = a/b$ are the solutions of (4.11). Indeed, one can check that (4.11) can be obtained from (3.24) by multiplication on $2\hbar z$ (see (3.22), (3.24)).

In (4.11) we can accomplish the limiting procedure to the equations (3.20) (or (3.19)) taking corresponding limits $a/b \equiv \lambda \to 0$ (or $\lambda \to \infty$).

From (3.26) we see that the functions

$$\Psi_{\lambda,n}(S, \varphi) = \left(\frac{\hbar}{\pi\lambda}\right)^{1/4} \exp\left(-\frac{(S - n\hbar)^2}{2\lambda\hbar}\right) \exp(in\varphi) \tag{4.12}$$

form the basis for the physical states (4.11). This basis satisfies the following ortho-normality conditions

$$\langle\Psi_{\lambda,n}|\Psi_{\lambda,m}\rangle \equiv \int \frac{dS d\varphi}{2\pi\hbar} \Psi_{\lambda,n}^*(S, \varphi)\Psi_{\lambda,m}(S, \varphi) = \delta_{nm}$$

With suitable normalization these basis functions have the limits as $\lambda \to 0$ (or $\lambda \to \infty$) in the dual space $\mathcal{L}_2^*(\mathcal{R}^1 \times \mathcal{S}^1)$ (see Appendix A). Indeed, the limit $\lambda \to 0$ of the function

$$\tilde{\Psi}_{\lambda,n}(S, \varphi) = \frac{1}{\sqrt{2\hbar}}\left(\frac{1}{\pi\lambda}\right)^{1/4} \Psi_{\lambda,n}(S, \varphi)$$

is the generalized function (3.21) which is a well defined linear functional on $\mathcal{L}_2(\mathcal{R}^1 \times \mathcal{S}^1)$. According to the rule (2.11) physical states (3.21) with different n form the ortho-normal basis of the corresponding reduced Hilbert space. Similarly, we can take the limit $\lambda \to \infty$ for the functions

$$\tilde{\tilde{\Psi}}_{\lambda,n}(S, \varphi) = \frac{1}{\sqrt{2}}\left(\frac{\lambda}{\pi\hbar}\right)^{1/4} \Psi_{\lambda,n}(S, \varphi)$$

and obtain

$$\lim_{\lambda\to 0} \tilde{\tilde{\Psi}}_{\lambda,n}(S, \varphi) = \psi_n(\varphi) = 1/\sqrt{2\pi} \exp(in\varphi)$$

This is the basis of the Hilbert space of canonical quantization (see (3.16)), and we have the same ortho-normality conditions due to the rule (2.11).

Obtained physical wave functions have other remarkable properties with respect to the described limiting procedure.

Let $\Psi_\lambda(S,\varphi)$ be any physical state (4.11) with unit norm. Then,

$$\Psi_\lambda(S,\varphi) = \sum_{n=-\infty}^{\infty} c_n \Psi_{\lambda,n}(S,\varphi) \qquad \text{with} \qquad \sum_{n=-\infty}^{\infty} |c_n|^2 = 1 \qquad (4.13)$$

where $\Psi_{\lambda,n}(S,\varphi)$ is the basis (4.12). If we integrate by φ the square of modulus of this function, and then take the limit $\lambda \to 0$, we obtain

$$\lim_{\lambda \to 0} \int \frac{d\varphi}{2\pi\hbar} \, |\Psi_\lambda(S,\varphi)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(S - n\hbar) \qquad (4.14)$$

We see that the right hand side of (4.14) describes the distribution function for the measurement of angular momentum S in the state Ψ_λ.

The same we can obtain for the normalized physical states (3.12) of Example 1. Namely,

$$\lim_{\epsilon \to 0} \int \frac{dp}{2\pi\hbar} \, |\langle p,q;\epsilon|\Psi\rangle|^2 = |\psi(q)|^2 \qquad (4.15)$$

where we use the representations (3.12) and (3.13′).

It is interesting to note that the integrands in (4.14) and (4.15) have similar properties. Indeed, using that

$$\lim_{\lambda \to 0} \Psi_{\lambda,n}^*(S,\varphi)\Psi_{\lambda,m}(S,\varphi) = 0, \qquad \text{when} \qquad m \neq n \qquad (4.14′)$$

we get

$$\lim_{\lambda \to 0} |\Psi_\lambda(S,\varphi)|^2 = 2\pi\hbar \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(S - n\hbar)$$

For Example 1, of course the integrand in (4.15) has zero limit (when $\epsilon \to 0$), since it is integrable on the plane and in this limit it does not depend on momentum p. If we neglect this zero factor we get the coordinate distribution function $|\psi(q)|^2$ (see 3.14).

These properties we use for the physical interpretation of wave functions Ψ_{ph} in Section 6, and now we return to the conditions (2.9) and minimization of $U_1(\Psi)$ for further investigation.

5 Minimal Uncertainties and Coherent States

We can consider the minimization principle for quadratic fluctuations using the functional $U_1(\Psi)$ (see (4.7)) as well. In this case instead of (4.9) we get the equation (see (C.10))

$$\frac{1}{2a^2}\hat{\Phi}_f^2|\Psi_{ph}\rangle + \frac{1}{2b^2}\hat{\Phi}_g^2|\Psi_{ph}\rangle - \frac{\hat{A}}{A}|\Psi_{ph}\rangle = 0 \qquad (5.1)$$

where $\hat{A}$ is a commutator (4.7), A is a parameter, and solutions $|\Psi_{ph}\rangle$ should satisfy (4.10) and the condition $\langle\Psi_{ph}|\hat{A}|\Psi_{ph}\rangle = A$ as well (see Appendix C).

There is some relation between the minimization of the functional $U_1(\Psi)$ and the condition (2.9). In our notations the condition (2.9) has the form

$$(\hat{\Phi}_f + i\epsilon\hat{\Phi}_g)|\Psi_\epsilon\rangle = 0 \qquad (5.2)$$

and for the wave function $\Psi_\epsilon(\xi)$ this is the first order differential equation. Of course, it is much easier to analyze solutions of (5.2) [11], then to investigate (4.9) (or (5.1)), which are the second order equations with two (or three) free parameters and subsidiary conditions (4.10). But, to be acceptable for the physical states, the corresponding solutions of (5.2) should belong to the domain of definition of self-adjoint operators $\hat{\Phi}_f$ and $\hat{\Phi}_g$. Except finiteness of the norm of $|\Psi_\epsilon\rangle$, this means that the operators $\hat{\Phi}_f$ and $\hat{\Phi}_g$ must be Hermitian on these functions. As it was pointed out, in general, these conditions are not fulfilled, and in that case we have to use the minimization principle for quadratic fluctuations of quantum constraints. But, if for some real ϵ, solutions of (5.2) satisfy the two conditions mentioned above, then we have (see Appendix C)

$$\langle\Psi_\epsilon|\hat{\Phi}_f^2|\Psi_\epsilon\rangle = \frac{\hbar\epsilon A}{2} \qquad \langle\Psi_\epsilon|\hat{\Phi}_g^2|\Psi_\epsilon\rangle = \frac{\hbar A}{2\epsilon} \qquad \text{where} \qquad \langle\Psi_\epsilon|\hat{A}|\Psi_\epsilon\rangle = A$$

and corresponding physical states $|\Psi_\epsilon\rangle$ provide minimization of the functional $U_1(\Psi)$: $U_1(\Psi_\epsilon) = \hbar^2/4$. Note (and it is natural) that such functions $|\Psi_\epsilon\rangle$ satisfy (5.1) ($|\Psi_{ph}\rangle = |\Psi_\epsilon\rangle$), with $a^2 = \hbar\epsilon A/2$, $b^2 = \hbar A/2\epsilon$ and $A = \langle\Psi_\epsilon|\hat{A}|\Psi_\epsilon\rangle$. To be convinced, it is sufficient to act on (5.2) by the operator $\hat{\Phi}_f - i\epsilon\hat{\Phi}_g$.

When the commutator $\hat{A}$ in (5.1) is a c-number, then (5.1) and (4.9) are equivalent and they define the same physical Hilbert spaces as the subspaces of $\mathcal{L}_2(\mathcal{M})$. But, in general, for given observables f and g these subspaces are different and to understand which one is more suitable further investigation is required. On the other hand, the functionals U and U_1 (and corresponding reduced physical Hilbert spaces) essentially depend on the choice of the pair of observables f, g. It turns out that reduced physical Hilbert spaces obtained by minimization of the functionals U and U_1 can be the same, even if the pair of observables f, g for U and U_1 are different. For example, physical states (3.25)-(3.26) were obtained from (5.2) with $f \equiv \tilde{f}_1$, $g \equiv \tilde{f}_a$ and $\epsilon = 1$ (see (3.22)-(3.23)). Respectively, these solutions minimize the functional $U_1(\Psi)$. In section 4 it was checked, that the same physical states minimize $U(\Psi)$ as well, but for $U(\Psi)$ the functions f and g are different ($f \equiv S$, $g \equiv \varphi$ (see (4.11)).

Let us return again to the choice of physical states by condition (2.9) (or (5.2)). For simplicity we consider the two dimensional case.

Suppose that solutions of (5.2) for some real ϵ satisfy the two required conditions, and hence, they are acceptable for the physical states. In complex variables $z = f - i\epsilon g$, $z^* = f + i\epsilon g$ condition (5.2) can be written as $\hat{\Phi}_{z^*}|\Psi_{ph}\rangle = 0$. The corresponding differential equation has the form (see (3.10') and (3.24))

$$\left(\partial_z - \frac{i}{\hbar}\theta_z(z, z^*)\right)\Psi_{ph}(z, z^*) = 0 \qquad (5.3)$$

---

[11]Practically it is always integrable.

where θ_z is the component of the 1-form $\theta = \theta_z dz + \theta_{z^*} dz^*$. Solutions of (5.3) are

$$\Psi_{ph}(z, z^*) = \exp\left(-\frac{1}{2}S(z, z^*)\right)\psi(z^*) \tag{5.4}$$

with arbitrary[12] $\psi(z^*)$, and $S = 2i/\hbar \int dz \theta_z$. These functions define the physical Hilbert space $\mathcal{H}_\epsilon$.

The pre-quantization operator $\hat{R}_{z^*} = z^* - \hat{\Phi}_{z^*}$ acts invariantly on the physical states (5.4), and this action is given by the multiplication of corresponding wave functions $\psi(z^*)$ by z^*

$$\hat{R}_{z^*}\Psi_{ph}(z, z^*) = z^*\Psi_{ph}(z, z^*) \tag{5.5}$$

From (2.7′) and (2.5) we have

$$[\hat{R}_z, \hat{\Phi}_{z^*}] = -i\hbar\hat{\Phi}_{\{z, z^*\}} = 2\epsilon\hbar\partial_z(\{f, g\})\hat{\Phi}_z + 2\epsilon\hbar\partial_{z^*}(\{f, g\})\hat{\Phi}_{z^*}$$

and if the Poisson bracket $\{f, g\}$ is not a constant, then the Physical Hilbert space (5.4) is not invariant under the action of pre-quantization operator $\hat{R}_z$. In this case, the deformation procedure is problematic (see Appendix B), and to define the operator $\hat{z}$ we use the relation between z, z^* variables. Since the operator $\hat{z}^+ \equiv \hat{R}_{z^*}$ is well defined on the physical states (5.4), it is natural to introduce the operator $\hat{z}$ as Hermitian conjugated to $\hat{R}_{z^*}$: $\hat{z} \equiv (\hat{R}_{z^*})^+$. Respectively, operators $\hat{f}$ and $\hat{g}$ will be

$$\hat{f} = \frac{1}{2}(\hat{z} + \hat{z}^+) \qquad \hat{g} = \frac{i}{2\epsilon}(\hat{z} - \hat{z}^+) \tag{5.6}$$

If $\Psi_{ph,n}(z, z^*)$ is some ortho-normal basis of the physical Hilbert space (5.4), then the action of the operator $\hat{z}$ on any state $\Psi_{ph}(z, z^*)$ can be written as

$$\begin{aligned}
\hat{z}\Psi_{ph}(z, z^*) &= \sum_n \Psi_{ph,n}(z, z^*)\langle\Psi_{ph,n}|\hat{z}|\Psi_{ph}\rangle = \sum_n \Psi_{ph,n}(z, z^*)\langle\hat{z}^+\Psi_{ph,n}|\Psi_{ph}\rangle = \\
&= \sum_n \Psi_{ph,n}(z, z^*)\int d\mu\ \Psi_{ph,n}^*(z', z'^*)z'\Psi_{ph}(z', z'^*) \tag{5.7}
\end{aligned}$$

where $d\mu \equiv d\mu(z', z'^*)$ is the standard measure (1.12).

Let us introduce the wave function $\chi_\zeta(z, z^*)$:

$$\chi_\zeta(z, z^*) \equiv \sum_n \Psi_{ph,n}^*(\zeta, \zeta^*)\Psi_{ph,n}(z, z^*) \tag{5.8}$$

Here ζ is considered as a complex parameter, and can take values in the same domain as the variable z. So, (5.8) is an expansion of the wave function $\chi_\zeta(z, z^*)$ in the basis $\Psi_{ph,n}(z, z^*)$ with coefficients $\Psi_{ph,n}^*(\zeta, \zeta^*)$.

With some assumptions about the analytical structure on $\mathcal{M}$ one can prove (see [10] and [13]), that the function $\chi_\zeta(z, z^*)$ is well defined, it is square integrable

$$\int d\mu\ |\chi_\zeta(z', z'^*)|^2 < \infty$$

---

[12]Class of holomorphic functions $\psi(z^*)$ should provide a finite norm of physical states $\Psi_{ph}(z, z^*)$

and the corresponding norm

$$\rho_\zeta^2 \equiv \int d\mu \ |\chi_\zeta|^2 = \sum_n \Psi_{ph,n}^*(\zeta, \zeta^*)\Psi_{ph,n}(\zeta, \zeta^*) = \chi_\zeta(\zeta, \zeta^*) \tag{5.8$'$}$$

does not depend on the choice of the basis $\Psi_{ph,n}(\zeta, \zeta^*)$.

Then, for an arbitrary physical state $|\Psi_{ph}\rangle$, (5.8) yields

$$\langle\chi_z|\Psi_{ph}\rangle = \int d\mu \ \chi_z^*(z', z'^*)\Psi_{ph}(z', z'^*) = \Psi_{ph}(z, z^*) \tag{5.9}$$

If we act with the operator $\hat{z}$ on the state $\chi_\zeta(z, z^*)$, and use (5.9) and (5.5), we obtain

$$\hat{z}\chi_\zeta(z, z^*) = \langle\chi_z|\hat{z}|\chi_\zeta\rangle = \langle\chi_\zeta|\hat{z}^+|\chi_z\rangle^* = (\hat{z}^+\chi_z(\zeta, \zeta^*))^* =$$
$$(\zeta^*\chi_z(\zeta, \zeta^*))^* = \zeta \ \chi_\zeta(z, z^*) \tag{5.10}$$

where in the last stage we take into account that

$$\chi_z^*(\zeta, \zeta^*) = \chi_\zeta(z, z^*)$$

which is apparent from the definition (5.8).

Thus, we see that the function $\chi_\zeta(z, z^*)$ is the eigenstate of the operator $\hat{z}$ with the eigenvalue ζ. This state is uniquely characterized by the complex parameter ζ. Sometimes we omit the coordinates of the phase space as the arguments of corresponding functions, and denote the state $\chi_\zeta(z, z^*)$ by χ_ζ, or $|\zeta\rangle$. We use also the notation $|\zeta\rangle \equiv |\bar{f}, \bar{g}; \epsilon\rangle$, where $\bar{f}$ and $\bar{g}$ are the real and imaginary parts of the complex number ζ respectively. From (5.8-10) we have the following properties of $|\zeta\rangle$ states

$$\int d\mu(\zeta) \ |\zeta\rangle\langle\zeta| = \hat{I} \tag{5.11}$$

$$\langle z|\zeta\rangle = \chi_\zeta(z, z^*) \tag{5.11$'$}$$

$$\hat{z}|\zeta\rangle = \zeta|\zeta\rangle \tag{5.11$''$}$$

It is remarkable, that the condition of completeness (5.11) allows us to introduce covariant and contravariant symbols of Berezin quantization [13].

Further, for the Hermitian operators (5.6) the relation (5.11$''$) takes the form

$$(\hat{f} - i\epsilon\hat{g})|\bar{f}, \bar{g}; \epsilon\rangle = (\bar{f} - i\epsilon\bar{g})|\bar{f}, \bar{g}; \epsilon\rangle \tag{5.12}$$

Then, we immediately get that

$$\langle\bar{f}, \bar{g}; \epsilon|\hat{f}|\bar{f}, \bar{g}; \epsilon\rangle = \bar{f} \qquad \langle\bar{f}, \bar{g}; \epsilon|\hat{g}|\bar{f}, \bar{g}; \epsilon\rangle = \bar{g} \tag{5.13}$$

and using the method described in Appendix C (for the details see [12]) we obtain

$$\frac{\langle\bar{f}, \bar{g}; \epsilon|(\hat{f} - \bar{f})^2|\bar{f}, \bar{g}; \epsilon\rangle\langle\bar{f}, \bar{g}; \epsilon|(\hat{g} - \bar{g})^2|\bar{f}, \bar{g}; \epsilon\rangle}{\langle\bar{f}, \bar{g}; \epsilon|\hat{C}|\bar{f}, \bar{g}; \epsilon\rangle^2} = \frac{\hbar^2}{4} \tag{5.14}$$

where $\hat{C}$ is the commutator $\hat{C} = i/\hbar \, [\hat{f}, \hat{g}]$. Note, that the operators $\hat{f}$ and $\hat{g}$ generally are not the pre-quantization ones, and respectively, commutator $\hat{C}$ is not of the form (1.18).

Thus, the quantum state $|\bar{f}, \bar{g}; \epsilon\rangle$ minimizes the quadratic fluctuations of the observables f and g around the values $\bar{f}$ and $\bar{g}$. In this respect they are very similar to the coherent states $|p, q, \epsilon\rangle$ (see (3.13)) which minimize the coordinate-momentum uncertainty.

For the considered examples (see Section 3) many technical calculations with coherent states can be accomplished explicitly. In case of plane the ortho-normal basis for the physical states (3.11) can be chosen as

$$\Psi_{ph,n}(z, z^*) = \exp\left(-\frac{1}{2}|z|^2\right) \frac{z^{*n}}{\sqrt{n!}} \tag{5.15}$$

Then, from (5.8) we get

$$\chi_\zeta(z, z^*) = \exp\left(-\frac{1}{2}|z|^2\right) \exp\left(-\frac{1}{2}|\zeta|^2\right) \exp\left(z^*\zeta\right) \tag{5.16}$$

and since $\chi_\zeta(\zeta, \zeta^*) = 1$, these states have the unit norm for arbitrary ζ (see (5.8′)). Comparing (5.11″) and (5.12) to (3.12) and (3.13) we see, that the states $|\zeta\rangle$ are just the usual coherent states $|p, q, \epsilon\rangle$ mentioned above.

For Example 2 we have

$$f = e^{S/\lambda} \cos\varphi \qquad g = e^{S/\lambda} \sin\varphi \qquad \epsilon = 1$$

and the complex variables z and z^* are given by (3.22). The physical Hilbert space is defined by (3.25), or (3.26), and we have the ortho-normal basis (4.12). Here, we omit the index "ph", arguments of the functions, and denote the corresponding basis by Ψ_n. The functions Ψ_n are the eigenstates of the operator $\hat{S} \equiv \hat{R}_S = -i\hbar\partial_\varphi$, with eigenvalues $n\hbar$. Then, from (5.8) and (4.12), for the states χ_z ($z = \exp(S/\lambda - i\varphi)$) we get

$$\chi_z = \left(\frac{\hbar}{\pi\lambda}\right)^{1/4} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(S - n\hbar)^2}{2\lambda\hbar}\right) \exp\left(in\varphi\right) \Psi_n \tag{5.17}$$

and this state has the norm

$$||\chi_z||^2 = \left(\frac{\hbar}{\pi\lambda}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(S - n\hbar)^2}{\lambda\hbar}\right) \tag{5.18}$$

which is obviously finite. In the limit $\lambda \to 0$ we obtain

$$||\chi_z||^2 \to \sum_{n=-\infty}^{\infty} \delta(S/\hbar - n) \tag{5.18′}$$

Let us introduce the operators $\hat{V}_\pm$

$$\hat{V}_\pm \Psi_n = \Psi_{n\pm1} \tag{5.19}$$

It is clear that the operators $V_{\pm}$ are equivalent to the phase operators $\exp(\pm i n\varphi)$ for the canonical quantization. From the definition (5.19) we have

$$\hat{V}_+\hat{V}_- = \hat{I} = \hat{V}_-\hat{V}_+ \qquad \hat{V}_+^+ = \hat{V}_- \tag{5.20}$$

Since the operator $\hat{z}^+$ acts as the multiplication on z^*, for the basis vectors (4.12) we get

$$\hat{z}^+\Psi_n = \exp\left(\frac{\hbar n}{\lambda} + \frac{\hbar}{2\lambda}\right)\Psi_{n+1}$$

Then, using the operator $\hat{V}_+$, we can represent the operator $\hat{z}^+$ in two different forms

$$\hat{z}^+ = \exp\left(\frac{\hat{S}}{\lambda} - \frac{\hbar}{2\lambda}\right)\hat{V}_+ = \hat{V}_+\exp\left(\frac{\hat{S}}{\lambda} + \frac{\hbar}{2\lambda}\right) \tag{5.21}$$

where we use that the basis vectors Ψ_n are the eigenvectors of the operator $\hat{S}$ with the eigenvalue $\hbar n$. Respectively, the Hermitian conjugated operator $\hat{z}$ is

$$\hat{z} = \hat{V}_-\exp\left(\frac{\hat{S}}{\lambda} - \frac{\hbar}{2\lambda}\right) = \exp\left(\frac{\hat{S}}{\lambda} + \frac{\hbar}{2\lambda}\right)\hat{V}_- \tag{5.22}$$

and, using (5.20), we obtain the commutator

$$[\hat{z}, \hat{z}^+] = 2\exp(2\hat{S}/\lambda)\sinh(\hbar/\lambda) \tag{5.23}$$

Note, that the corresponding classical commutation relation is

$$\{\hat{z}, \hat{z}^+\} = \frac{2i}{\lambda}\exp(2S/\lambda) \tag{5.24}$$

Now, from (5.22) and (5.17), we can check that the states χ_z are the eigenstates of the operator $\hat{z}$ with the eigenvalues $z = \exp(S/\lambda - i\varphi)$.

The states χ_z in (5.17) are defined for arbitrary value of the variable S. At the same time, the states with fixed value of the angular momentum ($\Delta S = 0$) exist only for the discrete values of S ($S = \hbar n$). Of course, the states χ_z are not the eigenstates of the operator $\hat{S}$, but, from (5.14), it is expected that $\Delta S \to 0$, when $\lambda \to 0$. Therefore, it is interesting to investigate the behavior of the states χ_z, when $\lambda \to 0$.

Note, that expansion (5.17) can be considered as the definition of the states χ_z for a quantum theory of a rotator in abstract Hilbert space; only the basis vectors Ψ_n should be the eigenstates of the angular momentum operator $\hat{S}$ with the eigenvalues $S_n = \hbar n$. With this remark we can neglect the dependence on the parameter λ in the basis vectors Ψ_n, and consider behavior (when $\lambda \to 0$) of corresponding coefficients only. If we introduce the vector $|S, \varphi; \lambda\rangle$ with unit norm

$$|S, \varphi; \lambda\rangle \equiv \frac{\chi_z}{||\chi_z||} \tag{5.25}$$

26

then, from (5.17), we get

$$|S, \varphi; \lambda\rangle = \sum_{n=-\infty}^{\infty} \frac{d_n(S, \lambda)}{d(S, \lambda)} \exp\left(in\varphi\right) \Psi_n$$

where

$$d_n(S, \lambda) = \exp\left(-\frac{(S - n\hbar)^2}{2\lambda\hbar}\right) \qquad d^2(S, \lambda) = \sum_{n=-\infty}^{\infty} d_n^2 \qquad (d > 0)$$

In the limit $\lambda \to 0$, $d_n(S, \lambda)/d(S, \lambda) \to c_n(S)$, and for the coefficients $c_n(S)$ we get:
a. $c_n(S) = 0$, if $S < \hbar(n - 1/2)$, or $S > \hbar(n + 1/2)$;
b. $c_n(S) = 1/\sqrt{2}$, if $S = \hbar(n - 1/2)$, or $S = \hbar(n + 1/2)$;
c. $c_n(S) = 1$, if $\hbar(n - 1/2) < S < \hbar(n + 1/2)$.
From this we obtain, that $|S, \varphi; \lambda\rangle \to \exp\left(in\varphi\right) \Psi_n$, where n is the nearest integer number to $S/\hbar$. But if $S/\hbar$ is exactly in the middle of two integers: $S/\hbar = n + 1/2$, then $|S, \varphi; \lambda\rangle \to 1/\sqrt{2}\ (\exp\left(in\varphi\right)\Psi_n + \exp\left(i(n + 1)\varphi\right)\Psi_{n+1})$. So, when $\lambda \to 0$, all states $|S, \varphi; \lambda\rangle$, with $\hbar(n - 1/2) < S < \hbar(n + 1/2)$, "collapse" to the state Ψ_n.

From (5.9) and (5.18′) we see that when $\lambda \to 0$, the behavior of the states $|S, \varphi; \lambda\rangle$ is equivalent to the corresponding behavior of the wave functions of E-quantization scheme given by (4.14″).

6   Quantum Distribution Functions and a Measurement Procedure

In this section we consider the physical interpretation of wave functions $\Psi_{ph}(\xi)$. For simplicity we refer again to the equation (5.2) and assume that the functions $f(\xi)$ and $g(\xi)$ are non-commuting observables ($\{f, g\} \neq 0$) on the two dimensional phase space $\mathcal{M}$. We assume as well that solutions of (5.2) $\Psi_\epsilon \equiv \Psi_{ph}(\xi)$ define the physical Hilbert space as the subspace of $\mathcal{L}_2(\mathcal{M})$. To emphasize dependence on the observables f, g and on the parameter ϵ, we denote this physical Hilbert space here by $\mathcal{H}_\epsilon(f, g)$.

On $\mathcal{H}_\epsilon(f, g)$ the operators $\hat{f}$ and $\hat{g}$ have the form (5.6), where the operator $\hat{z}^+$ acts on wave functions $\Psi_{ph}(\xi)$ as the multiplication by $z^*(\xi) = f(\xi) + i\epsilon g(\xi)$, and the operator $\hat{z}$ is it's Hermitian conjugated. Then, for mean values of these operators we get

$$\langle \Psi_{ph} | \hat{f} | \Psi_{ph} \rangle = \int d\mu(\xi)\ |\Psi_{ph}(\xi)|^2 f(\xi) \qquad \langle \Psi_{ph} | \hat{g} | \Psi_{ph} \rangle = \int d\mu(\xi)\ |\Psi_{ph}(\xi)|^2 g(\xi) \quad (6.1)$$

We see, that $|\Psi_{ph}(\xi)|^2$ can be interpreted as some "distribution function" on the phase space $\mathcal{M}$.

For further investigation we introduce the modulus and phase of wave functions $\Psi_{ph}(\xi)$

$$\Psi_{ph}(\xi) = e^{i\alpha(\xi)}\sqrt{\rho(\xi)} \tag{6.2}$$

27

From (5.2) and (6.2) we have two real equations [13]

$$V_f \alpha + \frac{\epsilon}{2} V_g (\log \rho) = \frac{1}{\hbar} \theta(V_f) \qquad V_g \alpha - \frac{1}{2\epsilon} V_f (\log \rho) = \frac{1}{\hbar} \theta(V_g) \qquad (6.3)$$

where V_f and V_g are the corresponding Hamiltonian vector fields (see (1.1)).

One can check a validity of the following relations

$$[V_f, V_g] = V_{\{f,g\}} = \frac{\{\{f,g\},g\}}{\{f,g\}} V_f - \frac{\{\{f,g\},f\}}{\{f,g\}} V_g$$

and

$$V_f \theta(V_g) - V_g \theta(V_f) = \{f,g\} + \theta(V_{\{f,g\}})$$

Using these relations, we can exclude the function $\alpha(\xi)$ from (6.3), and obtain the equation only for $\rho(\xi)$

$$\left[\frac{\hbar}{2\epsilon} \left(\frac{1}{\{f,g\}} V_f \right)^2 + \frac{\hbar\epsilon}{2} \left(\frac{1}{\{f,g\}} V_g \right)^2 \right] \log \rho = -\frac{1}{\{f,g\}} \qquad (6.4)$$

Note, that in variables f, g this equation takes the form

$$\frac{\hbar}{2} \left(\frac{1}{\epsilon} \partial_g^2 + \epsilon \partial_f^2 \right) \log \rho = -\frac{1}{\{f,g\}} \qquad (6.5)$$

where, the Poisson bracket $\{f,g\}$ can be considered as the function of f and g.

Any solution of (6.4) $\rho(\xi)$ defines corresponding phase $\alpha(\xi)$ up to the integration constant (see (6.3)). This constant phase factor is unessential for physical states (6.2), and respectively there is one to one correspondence between the "distribution functions" $\rho(\xi) = |\Psi_{ph}(\xi)|^2$ and the pure states described by a projection operator $\hat{P}_{\Psi_{ph}} \equiv |\Psi_{ph}\rangle\langle\Psi_{ph}|$

$$\rho(\xi) \longleftrightarrow \hat{P}_{\Psi_{ph}} \qquad (6.6)$$

With this remark we can use the index ρ for corresponding pure states as well: $\hat{P}_{\Psi_{ph}} \equiv \hat{P}_\rho$.

From (6.2) and (5.9) we have

$$\rho(\xi) = |\Psi(\xi)|^2 = \langle \chi_{z(\xi)} | \Psi_{ph} \rangle \langle \Psi_{ph} | \chi_{z(\xi)} \rangle \langle \chi_{z(\xi)} | \hat{P}_\rho | \chi_{z(\xi)} \rangle \qquad (6.7)$$

where $|\chi_{z(\xi)}\rangle$ is a coherent state related to the observables f and g (see (5.8), (5.12)). If one introduces the covariant symbol $P_\rho(\xi)$ of the projection operator $\hat{P}_\rho$

$$P_\rho(\xi) \equiv \frac{\langle \chi_{z(\xi)} | \Psi_{ph} \rangle \langle \Psi_{ph} | \chi_{z(\xi)} \rangle}{\langle \chi_{z(\xi)} | \chi_{z(\xi)} \rangle} \qquad (6.8)$$

then from (6.7) we have $\rho(\xi) = P_\rho(\xi) ||\chi_{z(\xi)}||^2$, and correspondence (6.6) describes well known connection between operators and their covariant symbols (see [13]).

---

[13] Note, that there are unessential singularities in the points ξ_α, where $\rho(\xi_\alpha) = 0$.

For any observable $F(\xi)$ one can introduce the corresponding operator $\hat{F}$ acting on the physical Hilbert space $\mathcal{H}_\epsilon(f, g)$, and standard quantum mechanical mean values are calculated by

$$\langle \Psi_{ph}|\hat{F}|\Psi_{ph}\rangle = Tr(\hat{F}\hat{P}_\rho) \equiv \langle \hat{F}\rangle_\rho \tag{6.9}$$

Let us introduce a new mean values $\bar{F}_\rho$:

$$\bar{F}_\rho \equiv \int d\mu(\xi)\ F(\xi)\rho(\xi) \tag{6.9'}$$

The connection between mean values $\langle \hat{F}\rangle_\rho$ and $\bar{F}_\rho$ generally is complicated and can be done only as an expansion in powers of $\hbar$. But, for $F = f$ and $F = g$ these mean values are the same for an arbitrary state ρ (see (6.1))

$$\bar{f}_\rho = \langle \hat{f}\rangle_\rho \qquad \bar{g}_\rho = \langle \hat{g}\rangle_\rho \tag{6.10}$$

Using again (5.6), for the operators $\hat{f}^2$ and $\hat{g}^2$ we obtain

$$\bar{f}^2_\rho = \langle \hat{f}^2\rangle_\rho + \frac{\epsilon\hbar}{2}\langle \hat{C}\rangle_\rho \qquad \bar{g}^2_\rho = \langle \hat{g}^2\rangle_\rho + \frac{\hbar}{2\epsilon}\langle \hat{C}\rangle_\rho \tag{6.11}$$

where the operator $\hat{C}$ is the commutator

$$\hat{C} = \frac{i}{\hbar}[\hat{f}, \hat{g}] \tag{6.12}$$

(see (5.14)).

Quadratic fluctuations calculated for the mean values (6.9) and (6.9') respectively are

$$(\Delta\hat{F})^2_\rho = \langle \hat{F}^2\rangle_\rho - \langle \hat{F}\rangle^2_\rho \tag{6.13}$$

and

$$(\Delta F)^2_\rho = \bar{F}^2_\rho - (\bar{F}_\rho)^2 \tag{6.13'}$$

Then, from (6.10) and (6.11) we have

$$(\Delta f)^2 = (\Delta\hat{f})^2 + \frac{\epsilon\hbar}{2}\langle \hat{C}\rangle \qquad (\Delta g)^2 = (\Delta\hat{g})^2 + \frac{\hbar}{2\epsilon}\langle \hat{C}\rangle \tag{6.14}$$

In general, a quantum system is not in a pure state and it is described by a density matrix operator $\hat{\rho}$ [14] which is Hermitian and semi-positive ($\langle \psi|\hat{\rho}|\psi\rangle \geq 0$, for any state $|\psi\rangle$), and it has the unit trace. Respectively, any density matrix operator has the spectral expansion

$$\hat{\rho} = \sum_n c_n|\psi_n\rangle\langle\psi_n| \tag{6.15}$$

where $|\psi_n\rangle$ are the ortho-normal eigenvectors of $\hat{\rho}$, c_n are the corresponding positive ($c_n > 0$) eigenvalues, and $\sum_n c_n = 1$.

Similarly to (6.7) we can introduce the "distribution function" $\rho(\xi)$ connected with the covariant symbol of $\hat{\rho}$

$$\rho(\xi) \equiv \langle \chi_{z(\xi)}|\hat{\rho}|\chi_{z(\xi)}\rangle \tag{6.16}$$

Using the spectral expansion (6.15) we get that a "distribution function" of a mixed state can be expressed as a convex combination of "distribution functions" of pure states

$$\rho(\xi) = \sum_n c_n \rho_n(\xi) \qquad (0 < c_n < 1) \tag{6.17}$$

One can easily check that the relations (6.10), (6.11) and (6.14) are valid for the mixed states as well.

From (6.16)-(6.17) we see that in general, a "distribution function" $\rho(\xi)$ is a smooth, non-negative function on the phase space $\mathcal{M}$, and it satisfies the standard condition of classical distributions

$$\int d\mu(\xi) \, \rho(\xi) = 1 \tag{6.18}$$

Note, that for the pure states the class of functions $\rho(\xi)$ essentially depend on the value of the parameter ϵ and on the choice of observables f and g. Indeed, for pure states this class is defined by solutions of equation (6.4), where this dependence is apparent. Therefore, sometimes it is convenient to indicate this dependence explicitly: $\rho(\xi) \equiv \rho(\xi|f, g; \epsilon)$. In the limit $\epsilon \to 0$ "distribution functions" $\rho(\xi|f, g; \epsilon)$ become singular (see (6.4)), and if the corresponding operator $\hat{f}$ has the discrete spectrum f_n, then in this limit functions $\rho(\xi|f, g; \epsilon)$ should collapse to the points of this spectrum f_n (see the end of Sections 4 and 5 and the remark below).

Thus, for a given $f(\xi)$, $g(\xi)$ and ϵ we have "distribution functions" $\rho(\xi|f, g; \epsilon)$ which look like classical ones, and at the same time they describe all possible quantum states uniquely. These functions form a convex set, and corresponding boundary points satisfy equation (6.4). Such functions $\rho(\xi|f, g; \epsilon)$ we call the quantum distribution functions.

We can compare a function $\rho(\xi|f, g; \epsilon)$ to a Wigner function $\rho_w(\xi)$, which is the Weil symbol of a density matrix operator [16]. For any Wigner function $\rho_w(\xi)$ we have the "classical" formula for quantum mechanical mean values

$$\langle \hat{F} \rangle_\rho = \int d\mu(\xi) \, F(\xi)\rho_w(\xi) \tag{6.19}$$

Though this formula is valid for an arbitrary observable $F(\xi)$, nevertheless Wigner functions can not be interpreted as a function of probability density. [14] In general, it is even negative in some domain of a phase space. It should be noted also, that a Wigner function is defined only for a "flat" phase space ($\mathcal{M} = \mathcal{R}^{2N}$) and cartesian coordinates.

A quantum distribution function $\rho(\xi|f, g; \epsilon)$ can be considered for almost arbitrary f, g "coordinates". It is always positive, but the "classical" formula (6.19) (with substitution ρ_w by ρ) is valid only for the functions $F = f$ and $F = g$.

For a physical interpretation of quantum distribution functions $\rho(\xi|f, g; \epsilon)$ we consider again Example 1 (see Section 3) with $\mathcal{M} = \mathcal{R}^2$, $f \equiv q$, $g \equiv -p$.

---

[14]Due to uncertainty principle there is no such function on the phase space of a quantum system.

In this case (6.14) takes the form

$$(\Delta q)^2 = (\Delta \hat{q})^2 + \frac{\epsilon \hbar}{2} \qquad (\Delta p)^2 = (\Delta \hat{p})^2 + \frac{\hbar}{2\epsilon} \tag{6.20}$$

where $(\Delta \hat{q})^2$ and $(\Delta \hat{p})^2$ are usual quantum mechanical quadratic fluctuations of coordinate and momentum. Since our quantum theory for any $\epsilon > 0$ is unitary equivalent to the coordinate (and momentum) representation, the quadratic fluctuations $(\Delta \hat{q})^2$ and $(\Delta \hat{p})^2$ can be calculated also by

$$(\Delta \hat{q})^2 = \int dq\ q^2 |\psi(q)|^2 - \left(\int dq\ q|\psi(q)|^2 \right)^2$$
$$(\Delta \hat{p})^2 = \int dp\ p^2 |\tilde{\psi}(p)|^2 - \left(\int dp\ p|\tilde{\psi}(p)|^2 \right)^2 \tag{6.21}$$

where $\psi(q)$ and $\tilde{\psi}(p)$ are wave functions of some pure state ρ in the coordinate and in the momentum representations respectively. The function $\tilde{\psi}(p)$ is the Fourier transformation of $\psi(q)$, and it is well known, that fluctuations (6.21) satisfy the Heisenberg uncertainty relation

$$(\Delta \hat{p})^2 (\Delta \hat{q})^2 \geq \frac{\hbar^2}{4} \tag{6.22}$$

From (6.20) we have

$$(\Delta q)^2 \geq \frac{\epsilon \hbar}{2}, \qquad (\Delta p)^2 \geq \frac{\hbar}{2\epsilon} \tag{6.23}$$

and using (6.22) we also get

$$(\Delta p)^2 (\Delta q)^2 \geq \hbar^2 \tag{6.24}$$

Let us introduce the functions

$$\rho_\epsilon(q) \equiv \int \frac{dp}{2\pi\hbar} \rho_\epsilon(p, q) \qquad \tilde{\rho}_\epsilon(p) \equiv \int \frac{dq}{2\pi\hbar} \rho_\epsilon(p, q) \tag{6.25}$$

where $\rho_\epsilon(p, q)$ is a quantum distribution function of this example. For pure states[15] $\rho_\epsilon(p, q)$ has the form (see (3.12))

$$\rho_\epsilon(p, q) = \langle p, q; \epsilon | \Psi_{ph} \rangle \langle \Psi ph | p, q; \epsilon \rangle \tag{6.26}$$

and using (3.13′) we obtain

$$\rho_\epsilon(q) = \left(\frac{1}{\pi\hbar\epsilon} \right)^{1/2} \int dq'\ \exp\left(-\frac{(q - q')^2}{\hbar\epsilon} \right) |\psi(q')|^2 \tag{6.27}$$

$$\tilde{\rho}_\epsilon(p) = \left(\frac{\epsilon}{\pi\hbar} \right)^{1/2} \int dp'\ \exp\left(-\frac{\epsilon(p - p')^2}{\hbar} \right) |\tilde{\psi}(p')|^2 \tag{6.27′}$$

In the limit $\epsilon \to 0$ and $\epsilon \to \infty$ we get

$$\text{when}\ \ \epsilon \to 0: \qquad \rho_\epsilon(q) \to |\psi(q)|^2, \qquad \tilde{\rho}_\epsilon(p) \to 0 \tag{6.28}$$

---

[15] Generalization to mixed states is straightforward.

31

$$\text{when} \quad \epsilon \to \infty : \qquad \rho_\epsilon(q) \to 0, \qquad \tilde{\rho}_\epsilon(p) \to |\tilde{\psi}(p)|^2 \qquad (6.28')$$

From definitions (6.21) and (6.25) we have the following correspondence between distribution functions and quadratic fluctuations

$$|\psi(q)|^2 \leftrightarrow (\Delta \hat{q})^2 \qquad |\tilde{\psi}(p)|^2 \leftrightarrow (\Delta \hat{p})^2 \qquad (6.29)$$

$$\rho_\epsilon(q) \leftrightarrow (\Delta q)^2 \qquad \tilde{\rho}_\epsilon(p) \leftrightarrow (\Delta p)^2 \qquad (6.29')$$

The function $|\psi(q)|^2$ is a probability density of coordinate distribution and, in principle, it can be measured. The corresponding experiment we denote by E_q. Theoretically it is assumed that in the experiment E_q the coordinate can be measured with the absolute precision, and a quantum system can be prepared in a given state as many times as it is necessary for a good approximation of the function $|\psi(q)|^2$. A statistical distribution of the coordinate, obtained in such experiment, is the intrinsic property of a quantum system in a given state: in general, in a pure state a definite value has some other observable (for example energy), but not the coordinate.

Similarly, for the momentum distribution function $|\tilde{\psi}(p)|^2$ we need the experiment E_p with a precise measurement of the momentum.

Thus, in the experiment E_q we can measure the distribution $|\psi(q)|^2$ and the corresponding quadratic fluctuation $(\Delta \hat{q})^2$:

$$E_q \longrightarrow |\psi(q)|^2 \longrightarrow (\Delta \hat{q})^2$$

and from the experiment E_p we get $|\tilde{\psi}(p)|^2$ and $(\Delta \hat{p})^2$:

$$E_p \longrightarrow |\tilde{\psi}(p)|^2 \longrightarrow (\Delta \hat{p})^2$$

One possible method for a measurement of a coordinate and a momentum of a quantum particle is a scattering of a light on this particle (see [14]). It is well known, that in such experiment the precise measurement of the coordinate can be achieved by photons with a very short wavelength λ (high energy). On the contrary, for the momentum measurement photons of low energy are needed. Theoretically, the experiment E_q is the measurements with photons of "zero wavelength": $\lambda \to 0$, and the experiment E_p requires photons of "zero energy": $\lambda \to \infty$. So E_q and E_p are two essentially different experiments. It should be noted, that in the experiment E_q we measure only the coordinate and we have no information about the momentum of a particle. Respectively, we have not any momentum distribution function for this experiment. Similarly, for the absolute precise measurements of the momentum, a particle can be in any point of the configuration space with equal to each other probabilities, and since the space is infinite, the coordinate distribution function vanishes.

But real experiments, of course, are with photons of finite and non-zero wavelength λ. Experiment with some fixed wavelength λ we denote by E_λ. In this experiment there is the error Δ_q in measuring of the coordinate and this error is proportional to the wavelength λ (see [14])

$$\Delta_q = \alpha \lambda \qquad (6.30)$$

Here α is a dimensionalless parameter of order 1 ($\alpha \sim 1$).

The momentum of a photon with a wavelength λ is

$$p_\lambda = \frac{2\pi\hbar}{\lambda}$$

and the error of momentum measurement is proportional to this momentum

$$\Delta_p = \beta\frac{\hbar}{\lambda} \tag{6.31}$$

where β is the parameter similar to α ($\beta \sim 1$).

Then, for the total quadratic fluctuations we can write

$$(\Delta_t q)^2 = (\Delta\hat{q})^2 + (\Delta_q)^2 = (\Delta\hat{q})^2 + \alpha^2\lambda^2$$
$$(\Delta_t p)^2 = (\Delta\hat{p})^2 + (\Delta_p)^2 = (\Delta\hat{p})^2 + \frac{\beta^2\hbar^2}{\lambda^2} \tag{6.32}$$

Thus, in the experiment E_λ we have two kind of fluctuations: the first one $((\Delta\hat{q}), (\Delta\hat{p}))$ is the intrinsic property of a quantum system, and the second $((\Delta_q), (\Delta_p))$ is related to the measurement procedure. As it is well known, the fluctuations $(\Delta\hat{q})$ and $(\Delta\hat{p})$ satisfy Heisenberg uncertainty principle (6.22). Assuming that for the ideal experiment $\alpha = \beta = 1/\sqrt{2}$, we can fix the uncertainties of a measurement procedure by

$$\Delta_q\Delta_p = \frac{\hbar}{2} \qquad \frac{\Delta_q}{\Delta_p} = \frac{\lambda^2}{\hbar} \tag{6.33}$$

With this assumption, from (6.20) and (6.32), we can write

$$(\Delta_t q)^2 = (\Delta q)^2 \qquad (\Delta_t p)^2 = (\Delta p)^2 \tag{6.34}$$

and the parameter ϵ and the wavelength λ are related by

$$\lambda = \sqrt{\epsilon\hbar} \tag{6.35}$$

Recall that the quadratic fluctuations $(\Delta q)^2$ and $(\Delta p)^2$ are calculated by the mean values of the function $\rho_\epsilon(p,q)$ (see (6.20)). Taking into account (6.32)-(6.35) one can suppose that these fluctuations respectively are the total quadratic fluctuations of the coordinate and the momentum measured in the experiment E_λ. As it was mentioned, in this experiment we have some unavoidable non-zero measurement error both for the coordinate and the momentum, and the parameter ϵ fixes the ratio of these errors (see (6.33), (6.35))

$$\epsilon = \frac{\Delta_q}{\Delta_p} \tag{6.35'}$$

It is worth noting that in the experiments E_λ one can carry out the separate measurement of coordinate and momentum as well as do it simultaneously. Then, the function $\rho_\epsilon(q)$ (see (6.27) and (6.29')) can be interpreted, as a distribution of the coordinate

obtained in the experiment E_λ. Similarly, the function $\tilde{\rho}_\epsilon(p)$ corresponds to the momentum measurements in E_λ. In the limit $\lambda \to 0$ ($\epsilon \to 0$) we get the experiment E_q with the coordinate distribution $\rho_0(q) = |\psi(q)|^2$ only, and in the opposite limit $\lambda \to \infty$ (the experiment E_p) only the distribution $|\tilde{\psi}(p)|^2$ remains (see (6.28))[16]

Now, it is natural to suppose that the quantum distribution function $\rho_\epsilon(p, q) = |\Psi_\epsilon(p, q)|^2$ is the distribution obtained in the experiment E_λ with simultaneous measurements of the coordinate and the momentum.

This idea can be easily generalized assuming that the quantum distribution function $\rho(\xi|f, g : \epsilon)$ is the distribution on the phase space obtained in some ideal experiment with simultaneous measuring of f and g observables. In such experiment we have the unavoidable errors Δ_f and Δ_g connected with the measurement procedure with micro-objects. For corresponding fluctuations there is the additional uncertainty principle (see (6.33)), and the parameter ϵ specifies the experiment by fixing the ratio of the errors $\epsilon = \Delta_f/\Delta_g$.

If the function $\rho(\xi) \equiv \rho(\xi|f, g; \epsilon)$ is really measurable, then in the limit $\epsilon \to 0$ this function $\rho(\xi)$ should describe the experimental distribution of the exact measurement of the observable f. It is obvious that for the observable f with discrete spectrum corresponding function $\rho(\xi)$ should be localized in the points of this spectrum. Thus, by asymptotics of quantum distribution functions one can obtain the spectrum of the physical observables (see (4.14'')).

We see that quantum distribution functions can play some fundamental role for the interpretation of quantum theory. It is natural to try to formulate quantum mechanics in terms of these distribution functions, especially as, they describe all possible states of a quantum system uniquely. But for this it is worthwhile to have an independent (without referring to the Hilbert space) description of the set of functions $\rho(\xi) \equiv \rho(\xi|f, g; \epsilon)$. Corresponding functions are positive, satisfying (6.18), and at the same time they essentially depend on the choice of observables f and g and of the parameter ϵ. On the other hand, the set of physical states is a convex one, were the boundary points are the pure states. So for the description of our set we need to specify the distribution functions of pure states, but the latter are given as the solutions of (6.4). Thus, in this approach the important role plays the equation (6.4). Actually it describes the set of all physical states and, respectively, it contains the information about quantum uncertainties both the intrinsic and the experimental ones.

Note, that on the left hand side of the corresponding equation there is the Laplace operator (see (6.4)-(6.5)) and we have some induced metric structure on the phase space $\mathcal{M}$. It is remarkable, that this metric structure is related to the experimental errors. Indeed, in case of Example 1 these errors are (see (6.33-(6.35))

$$\Delta_q = \sqrt{\frac{\epsilon\hbar}{2}} \qquad \Delta_p = \sqrt{\frac{\hbar}{2\epsilon}}$$

and it is easy to see that corresponding equation (6.5) takes the form

$$\left(\Delta_q^2 \partial_q^2 + \Delta_p^2 \partial_p^2\right) \log \rho = -1 \tag{6.36}$$

---

[16]The distributions $\rho_\infty(q)$ and $\tilde{\rho}_0(p)$ are degenerated to zero functions.

Such kind of phase space "shadow" metric was introduced in [4c].

If the equation for the quantum distribution functions of pure states has really the fundamental character, then one might expect that it can be derived from some general principle. A suitable principle could be the minimization of certain functional, and we arrive to the problem of construction of the corresponding functional. Since the minimization should be achieved on pure states, it is natural to interpret such functional as the entropy of a quantum system. Respectively, one candidate for such functional is the standard quantum mechanical entropy $S = -Tr(\hat{\rho}\log\hat{\rho})$ which can be expressed as the functional of $\rho(\xi)$.

It seems, that this and other above mentioned problems are interesting and need further investigation.

Appendix A

Let f, g be two non-commuting observables and $\hat{\Phi}_f, \hat{\Phi}_g$ the corresponding constraint operators (2.7). As it was mentioned, these operators are Hermitian on the Hilbert space $\mathcal{H} \equiv \mathcal{L}_2(\mathcal{M})$. Suppose, that the equation (see (2.9))

$$(\hat{\Phi}_f + i\epsilon\hat{\Phi}_g)|\Psi_\epsilon\rangle = 0 \tag{A.1}$$

has normalizable solutions for any $\epsilon \in (0, \delta)$, where δ is some positive number. The solutions with fixed ϵ form some subspace $\mathcal{H}_\epsilon$ of the Hilbert space $\mathcal{H}$. We assume, that each subspace can be represented as $\mathcal{H}_\epsilon = F_\epsilon \mathcal{H}_0$, where $\mathcal{H}_0$ is some linear space, and F_ϵ is a linear invertible map

$$F_\epsilon : \mathcal{H}_0 \to \mathcal{H}_\epsilon \qquad F_\epsilon^{-1} : \mathcal{H}_\epsilon \to \mathcal{H}_0 \tag{A.2}$$

In practical applications the linear space $\mathcal{H}_0$ automatically arises from the form of the general solution of (A.1); only it should be specified from the condition of square integrability of corresponding functions $\Psi_\epsilon = F_\epsilon \psi$, where $\psi \in \mathcal{H}_0$. For example, in case of eq. (3.3), the general solution (3.4)-(3.5) is described by the space of polynomials $P(\xi)$, and it can be interpreted as $\mathcal{H}_0$. The representation (3.12) and (3.15) of the same solutions is different, and in that case, the space $\mathcal{H}_0$ obviously is $\mathcal{L}_2(\mathcal{R}^1)$. As for the general solution (3.25)-(3.26), the space $\mathcal{H}_0$ is a space of Fourier modes c_n, $n \in Z$, with $\sum |c_n|^2 < \infty$ (see (3.27)).

The space of linear functionals on the Hilbert space $\mathcal{H}$ is called the dual (to $\mathcal{H}$) space, and we denote it by $\mathcal{H}^*$. From our definitions we have

$$\Psi_\epsilon = F_\epsilon \psi \in \mathcal{H}_\epsilon \subset \mathcal{H} \subset \mathcal{H}^*$$

Suppose that the set of vectors $F_\epsilon \psi$ with any fixed $\psi \in \mathcal{H}_0$, has the limit ($\epsilon \to 0$) in the dual space $\mathcal{H}^*$, and this limit defines the vector $\psi_* \in \mathcal{H}^*$

$$\lim_{\epsilon \to 0} F_\epsilon \psi = \psi_* \tag{A.3}$$

Such linear functional ψ_* usually is unbounded, and the limit in (A.3) means that for any $\Psi \in \mathcal{H}$ we have[17]

$$\lim_{\epsilon \to 0} \langle F_\epsilon \psi | \Psi \rangle = \psi_*(\Psi) \qquad (A.3')$$

where $\psi_*(\Psi)$ denotes the value of the functional ψ^* on the corresponding vector $\Psi \in \mathcal{H}$. If we change the map F_ϵ by

$$F_\epsilon \to \tilde{F}_\epsilon = a(\epsilon) F_\epsilon$$

where $a(\epsilon)$ is some "scalar" function of the parameter ϵ, then the new map $\tilde{F}_\epsilon$ provides representation of the subspace $\mathcal{H}_\epsilon$ in the same form: $\mathcal{H}_\epsilon = \tilde{F}_\epsilon \mathcal{H}_0$. It is obvious that existence of the limit in (A.3) essentially depends on the suitable choice of the normalizable function $a(\epsilon)$.

The action of some operator $\hat{O}$ on the functional ψ_* can be defined by

$$\hat{O}\psi_*(\Psi) \equiv \psi_*(\hat{O}^+\Psi) \qquad (A.4)$$

where $\hat{O}^+$ is the Hermitian conjugated to $\hat{O}$.

The norm $||\Psi_\epsilon||$ of the vectors $\Psi_\epsilon = F_\epsilon \psi$, with fixed ψ, usually diverges when $\epsilon \to 0$, but if we assume that

$$\epsilon ||F_\epsilon \psi|| \to 0 \qquad (A.5)$$

then we can prove that ψ_* satisfies the equation $\hat{\Phi}_f \, \psi_* = 0$. Indeed, from (A.3)-(A.5) we have

$$\hat{\Phi}_f \psi_*(\Psi) = \psi_*(\hat{\Phi}_f \Psi) = \lim_{\epsilon \to 0} \langle F_\epsilon \psi | \hat{\Phi}_f \Psi \rangle = \lim_{\epsilon \to 0} \langle \Psi_\epsilon | \hat{\Phi}_f \Psi \rangle = \lim_{\epsilon \to 0} i\epsilon \langle \Psi_\epsilon | \hat{\Phi}_g \Psi \rangle = 0 \quad (A.6)$$

where we take into account that the function $\Psi_\epsilon = F_\epsilon \psi$ satisfies (A.1). Thus, (A.3) defines the map $F_* : \mathcal{H}_0 \to \mathcal{H}^*$, and corresponding functionals $\psi_* = F_* \psi$ satisfy condition (2.8).

Further, let us assume, that $F_* \psi \neq 0$, whenever $\psi \neq 0$. Then, the space $\mathcal{H}_{ph} \equiv F_* \mathcal{H}_0$, as the linear space, will be isomorphic to $\mathcal{H}_0$, and, respectively, isomorphic to each $\mathcal{H}_\epsilon$ as well (see (A.2)).

If for $\forall \, \epsilon_1, \epsilon_2 \in (0, \delta)$ the map

$$F_{\epsilon_2} F_{\epsilon_1}^{-1} : \mathcal{H}_{\epsilon_1} \to \mathcal{H}_{\epsilon_2} \qquad (A.7)$$

is an unitary transformation, then one can introduce the Hilbert structure on $\mathcal{H}_0$ and $\mathcal{H}_{ph}$ by definition of the scalar product

$$\langle \psi_2 | \psi_1 \rangle \equiv \langle F_* \psi_2 | F_* \psi_1 \rangle \equiv \langle F_\epsilon \psi_2 | F_\epsilon \psi_1 \rangle \qquad (A.8)$$

It is obvious that in case of unitarity of transformations (A.7) the scalar product (A.8) is independent on the choice of the parameter ϵ, and the corresponding Hilbert structure is a natural. But, in general, transformation (A.7) is not the unitary one, and there is no some special Hilbert structure on $\mathcal{H}_0$. Respectively, we have the problem

---

[17]As an unbounded functional ψ_* is not defined for an arbitrary $\Psi \in \mathcal{H}$, but the domain of definition of ψ_* should be everywhere dense set in $\mathcal{H}$.

for the scalar product on the space $\mathcal{H}_{ph}$, especially as, corresponding functionals are unbounded and have the "infinite norm" in the Hilbert space $\mathcal{H}$.

Note, that for the general solutions (3.4)-(3.5) corresponding transformation (A.7) is not the unitary one, while the general solution (3.11)-(3.12), (3.15) provides unitarity explicitly

$$\tilde{\Psi}_{\epsilon_2}(p,q) = \int \frac{dpdq}{2\pi\hbar} \, \langle p,q;\epsilon_2|p',q';\epsilon_1\rangle \tilde{\Psi}_{\epsilon_1}(p',q')$$

Now, we describe some procedure for the solution of scalar product problem in that general case too.

In ordinary quantum mechanics a physical state is represented by a ray in a Hilbert space, and all vectors on the same ray are physically indistinguishable. So, if we suppose that the vector $|\psi_*\rangle$ has some norm $||\psi_*||$, then the normalized vector

$$|\psi_*\rangle\rangle \equiv \frac{|\psi_*\rangle}{||\psi_*||} \tag{A.9}$$

describes the same physical state. It is just the scalar product of such normalized vectors that has the physical meaning. Up to the phase factor, this scalar product describes the "angle" between the rays, and defines the probability amplitude.

We introduce the scalar product of such normalized vectors by

$$\langle\langle\psi_{2*}|\psi_{1*}\rangle\rangle \equiv \lim_{\epsilon\to 0} \frac{\langle\Psi_{2\epsilon}|\Psi_{1\epsilon}\rangle}{||\Psi_{2\epsilon}|| \, ||\Psi_{1\epsilon}||} \tag{A.10}$$

where the limits of $|\Psi_{1\epsilon}\rangle$ and $|\Psi_{2\epsilon}\rangle$ respectively are the functionals $|\psi_{1*}\rangle$ and $|\psi_{2*}\rangle$ (see (A.3)), and the latter are related to $|\psi_{1*}\rangle\rangle$ and $|\psi_{2*}\rangle\rangle$ by (A.9). When the limit (A.10) exists, it should define the scalar product of the normalized physical states. Then, the scalar product for arbitrary vectors can be obtained uniquely up to a rescaling.

It is obvious, that in case of unitarity of transformations (A.7), the definitions of scalar product (A.8) and (A.10) are equivalent.

Note, that the described scheme for the definition of scalar product of physical states (2.8) can be generalized for other constrained systems as well.

Appendix B

Let us consider a symplectic manifold $\mathcal{M}$ with global coordinates ξ^k, $(k = 1, ..., 2N)$ and constant symplectic matrix: $\partial_j\omega^{kl} = 0$, where $\omega^{kl} = -\{\xi^k, \xi^l\}$ (see (1.2′)). The simple example of such $\mathcal{M}$ is $\mathcal{R}^{2N}$ with canonical coordinates.

For the global coordinates ξ^k we can introduce the corresponding constraint functions Φ_{ξ^k}, and from (1.6′)-(1.7) we get

$$\Phi_{\xi^k} = \Phi^k = \omega^{kl}(P_l - \theta_l) \tag{B.1}$$

Then, (1.8) takes the form

$$\{\Phi^k, \Phi^l\}_* = \omega^{kl} \qquad \{f, \Phi^k\}_* = -\omega^{kl}\partial_l f \tag{B.2}$$

where $f(\xi)$ is any observable on $\mathcal{M}$, but in (B.2) it is considered as a function on $T^*\mathcal{M}$ with natural extension (see remarks after eq. (1.8)).

Let us add to the function $f(\xi)$ the term linear in constraints Φ^k

$$f(\xi) \to f^{(1)} = f(\xi) + A_l^{(1)}(\xi)\Phi^l \tag{B.3}$$

and choose the functions $A_l^{(1)}(\xi)$ to satisfy the condition

$$\{f^{(1)}, \Phi^k\}_* = B_l^{(1)k}(\xi)\Phi^l \tag{B.4}$$

This means, that the right hand side of (B.4) should contain the constraints Φ^k only in the first degree. From this condition the functions $A_l^{(1)}(\xi)$ and $B_l^{(1)k}(\xi)$ are defined uniquely

$$A_l^{(1)}(\xi) = -\partial_l f(\xi) \qquad B_l^{(1)k}(\xi) = \omega^{kj}\partial_{jl}^2 f(\xi) \tag{B.5}$$

It is obvious, that $f^{(1)} = R_f$, and (B.4)-(B.5) are equivalent to (2.2) and (2.5) with constant symplectic matrix ω^{kl}. We can continue this "deformation" procedure

$$f^{(1)} \to f^{(2)} = f^{(1)} + \frac{1}{2}A_{lj}^{(2)}(\xi)\Phi^l\Phi^j \tag{B.6}$$

demanding

$$\{f^{(2)}, \Phi^k\}_* = B_{lj}^{(2)k}(\xi)\Phi^l\Phi^j$$

Then, for the functions $A_{lj}^{(2)}(\xi)$ and $B_{lj}^{(2)k}(\xi)$ we have

$$A_{lj}^{(2)}(\xi) = \partial_{lj}^2 f(\xi) \qquad B_{lj}^{(2)k}(\xi) = -\frac{1}{2}\omega^{ki}\partial_{ilj}^3 f(\xi)$$

Generalizing for arbitrary n, we get

$$f(\xi) \to f^{(n)} = f(\xi) + \sum_{a=1}^{n} \frac{1}{a!}A_{k_1...k_a}^{(a)}(\xi)\Phi^{k_1}....\Phi^{k_a} \tag{B.7a}$$

where

$$A_{k_1...k_a}^{(a)}(\xi) = (-)^a \partial_{k_1...k_a}^{(a)} f(\xi) \tag{B.7b}$$

and

$$\{f^{(n)}, \Phi^k\}_* = \frac{(-)^{n+1}}{n!}\omega^{kl}\left(\partial_{lk_1...k_n}^{(n+1)} f(\xi)\right)\Phi^{k_1}....\Phi^{k_n} \tag{B.7c}$$

Using this procedure for any observable $f(\xi)$, one can construct a new function $\tilde{f} = \lim f^{(n)}$ $(n \to \infty)$, which commutes with all constraints Φ^k $(k = 1, ..., 2N)$, and on the constraint surface $(\Phi^k = 0$ $(k = 1, ..., 2N))$ it is equal to $f(\xi)$.

A similar procedure can be accomplished on the quantum level as well, taking into account operators ordering and self-adjoint conditions. But, when the symplectic matrix ω^{kl} depends on coordinates ξ^k, the described procedure fails for some observables $f(\xi)$, even on the classical level. For the illustration let us consider a simple example on a half plane with coordinates (p, q), $p > 0$, and the canonical 1-form $\theta = p\,dq$. If we take the coordinates $\xi^1 = p^2/2$, $\xi^2 = q$ (which are global here), then the corresponding constraints $\Phi^1 = p^2 - pP_q$, $\Phi^2 = P_p$ have the commutation relations

$$\{\Phi^2, \Phi^1\} = p + \frac{1}{p}\Phi^1 \tag{B.8}$$

The first deformation of the function $f = q$, as usual, gives $f^{(1)} = R_q = q - P_p$ and we get

$$\{f^{(1)}, \Phi^1\}_* = -\frac{1}{p}\Phi^1 \qquad \{f^{(1)}, \Phi^2\}_* = 0 \tag{B.9}$$

Considering the second deformation (B.6)

$$f^{(2)} = f^{(1)} + \frac{1}{2}\left(A_{11}(\xi)(\Phi^1)^2 + 2A_{12}(\xi)\Phi^1\Phi^2 + A_{22}(\xi)(\Phi^2)^2\right)$$

and using commutation relations (B.8)-(B.9), we see, that it is impossible to cancel the linear (in constraints Φ^1 and Φ^2) terms in the Poisson brackets $\{f^{(2)}, \Phi^1\}_*$ and $\{f^{(2)}, \Phi^2\}_*$ simultaneously.

Appendix C

At first we consider minimization of the product of quadratic fluctuations (see (4.6))

$$U(\Psi) \equiv \langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle \tag{C.1}$$

with the vectors $|\Psi\rangle$ of unit norm

$$\langle\Psi|\Psi\rangle = 1 \tag{C.2}$$

For the minimization of the functional $U(\Psi)$ one can use the variation principle, considering the variation of $|\Psi\rangle$ to be independent of $\langle\Psi|$. Since we have the subsidiary condition (C.2), from the variation of (C.1) we obtain

$$b^2\hat{\Phi}_f^2|\Psi\rangle + a^2\hat{\Phi}_g^2|\Psi\rangle = c|\Psi\rangle \tag{C.3}$$

where

$$a^2 = \langle\Psi|\hat{\Phi}_f^2|\Psi\rangle \qquad b^2 = \langle\Psi|\hat{\Phi}_g^2|\Psi\rangle \tag{C.4}$$

Multiplying by $\langle\Psi|$, we get $c = 2a^2b^2$, and the equation (C.3) takes the form

$$\frac{1}{2a^2}\hat{\Phi}_f^2|\Psi\rangle + \frac{1}{2b^2}\hat{\Phi}_g^2|\Psi\rangle = |\Psi\rangle \tag{C.5}$$

Thus, the solutions of (C.5), which satisfy conditions (C.4) can provide minimization of the functional $U(\Psi)$. If there are solutions with different values of the parameters a and b, then we have to choose the solutions with minimal value of the product $a^2 b^2$.

Now we consider minimization of the functional $U_1(\Psi)$ (see (4.7))

$$U_1(\Psi) \equiv \frac{\langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle}{\langle\Psi|\hat{A}|\Psi\rangle^2} \tag{C.6}$$

For an arbitrary vector $|\Psi\rangle$ and any real parameter ϵ we have

$$\langle\Psi|(\hat{\Phi}_f - i\epsilon\hat{\Phi}_g)(\hat{\Phi}_f + i\epsilon\hat{\Phi}_g)|\Psi\rangle \geq 0 \tag{C.7}$$

The left hand side of this inequality is a second ordered polynomial in ϵ

$$\epsilon^2\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle - \hbar\epsilon\langle\Psi|\hat{A}|\Psi\rangle + \langle\Psi|\hat{\Phi}_f^2|\Psi\rangle$$

and respectively we have

$$\langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle \geq \frac{\hbar^2}{4}\langle\Psi|\hat{A}|\Psi\rangle^2 \tag{C.8}$$

Thus, the minimal value of the functional $U_1(\Psi)$ could be $\hbar^2/4$. If for some ϵ the equation

$$(\hat{\Phi}_f + i\epsilon\hat{\Phi}_g)|\Psi\rangle = 0 \tag{C.9}$$

has normalizable solution $|\Psi\rangle = |\Psi_\epsilon\rangle$, then, for this $|\Psi_\epsilon\rangle$ we have an equality in (C.7) and (C.8). Respectively, this states $|\Psi_\epsilon\rangle$, provide minimization of the functional $U_1(\Psi)$. But, as it was indicated in section 4, sometimes equation (C.9) has no normalizable solutions for any real ϵ. In that case, one can consider minimization problem for the functional $U_1(\Psi)$ by variation principle, as it was done above for the functional $U(\Psi)$. Repeating the same procedure, we get the equation

$$\frac{1}{2a^2}\hat{\Phi}_f^2|\Psi\rangle + \frac{1}{2b^2}\hat{\Phi}_g^2|\Psi\rangle - \frac{\hat{A}}{A}|\Psi\rangle = 0 \tag{C.10}$$

where a, b, A are parameters, and the solution $|\Psi\rangle$ should satisfy (C.4) and the additional condition $\langle\Psi|\hat{A}|\Psi\rangle = A$ as well.

Acknowledgments

References

[1] N.M.J. Woodhous, *Geometric Quantization*. Clarendon, Oxford, (1992).

[2] a) G. Jorjadze and I. Sarishvili, Theor. Math. Phys. **93**, 1239 (1993).
G. Jorjadze, G. Lavrelashvili and I. Sarishvili, J. Phys. A: Math. Gen. **26**, 751, (1993).

[3] a) P.A.M. Dirac, *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, Yeshive University, New York, (1964).
b) K. Sundermeyer, *Constrained Dynamics*. Lecture Notes In Physics, bf 169, Springer Verlag, Berlin - Heidelberg - New York, (1982).
c) M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*. Princeton University Press, Princeton, (1992).

[4] a) E.S. Fradkin, V.Ya. Linetski, Nucl. Phys. **B431**, 569, (1994).
E.S. Fradkin, V.Ya. Linetski, *BFV Quantization on Hermitian Symmetric Spaces*. Preprint IC/94/393.
b) E. Gozzi, Phys. Lett. **A, 202**, 330, (1995).
c) J.R. Klauder, *Geometric Quantization from a Coherent State Viewpoint*. (to appear in the proceedings of the "XIV Workshop on Geometric Methods in Physics", Bielowieza, Poland, July 9-15, 1995).
J.R. Klauder and P.Maraner, *Dynamics as Shadow of the Phase Space Geometry*. Preprint UPRF-96-462 (1996).

[5] A. Ashtekar and R.S. Tate, J. Math. Phys. **35** (12), 6434, (1994).

[6] L.D. Faddeev and S. Shatashvili, Phys. Lett. **B, 167**, 225 (1986).

[7] S, Gupta, Proc. R.Soc., Ser. A, **63**, 681, (1950).
K. Bleuler, Helv. Phys. Acta, **23**, 567, (1950).

[8] L.D. Faddeev, A.A. Slavnov, *Introduction to the Quantum Theory of Gauge Fields*, Moscow, Nauka, (1984).

[9] M. Stone, Proc. Nat. Acad. Sci. USA, **16**, 172, (1930).
J. von-Neumann, Math. Ann., **104**, 570, (1931).

[10] L.R. Klauder and B.S. Skagerstam, *Coherent States, Applications in Physics and Mathematical Physics*, World Sci. Publ. Co., Singapore, (1985).

A.M. Perelomov, *generalized Coherent States and Their Applications.* Berlin, Springer-Verlag, (1986).

[11] A.M. Perelomov, *Integrable systems of Classical Mechanics and Lie Algebras.* Birkhauser Verlag, Basel, (1990).

[12] R. Jackiw, J. Math. Phys., **9**, 339, (1968).
P. Carruthers and M. Nieto, Rev. Mod. Phys., **40**, 411, (1968).

[13] F.A. Berezin, Comm. Math. Phys. **40**, 153, (1975).

[14] J. von-Neumann, *Mathematical Foundations of Quantum Mechanics.* Princeton University Press, Princeton, (1955).

[15] E. Wigner, Phys. Rev. **40**, 740, (1932).